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A New Efficient Method for solving Helmholtz and Coupled Helmholtz Equations Involving Local Fractional Operators

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Abstract:

In this manuscript, we apply a new technique, namely local fractional Laplace variational iteration method (LFVITM) on Helmholtz and coupled Helmholtz equations to obtain the analytical approximate solutions. The iteration procedure is based on local fractional derivative operators (LFDOs). This method is the combined of the local fractional Laplace transform (LFLT) and variational iteration method (VIM). The method in general is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the new technique.

Keywords: Analytical approximate solutions; Helmholtz Equation; Local fractional Laplace variational iteration method.

1. Introduction:

In recent years, a many of approximate and analytical methods have been utilized to solve the partial differential equations with local fractional derivative operators such as the LFFSM, Hu et al. (2012), LFLDM, Hao ,et.al. (2013), LFVIM, Su et al. (2013), LFSEM, Yang et al. (2013). LFSTM, Srivastava et al. (2014) LFLTM, Zhao et al. (2014). LFFDM, Wang et al. (2014) and Yan et al. (2014). LFDM, Baleanu et al. (2016), LFDTM, Yang et al. (2016) and Jafari (2016) and LFRDTM, Jafari et al. (2016). and Our main aim in this work, we propose an efficient modification of the LFVIM to solve Helmholtz and coupled Helmholtz equations with local fractional derivative operators. It is important to note that the new technique reduces the size of calculations compared to the local fractional vibrational iteration method (LFVIM).

The Helmholtz equation with local fractional derivative operators in two-dimensional case was suggested as follows:

$$\frac{\partial^{2\alpha}H(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}H(x,y)}{\partial y^{2\alpha}}+\omega^{2\alpha}H(x,y)=f(x,y),\ 0<\alpha\le 1 \qquad (1.1)$$

with the initial value conditions as follows:

$$H(0,y)=\varphi(y),\quad \frac{\partial^{\alpha}H(0,y)}{\partial x^{\alpha}}=\psi(y) \qquad (1.2)$$

where $H(x,y)$ is unknown function and $f(x,y)$ is a source term.

This paper is organized as follows: In Section 2, we recall the local fractional calculus (LFC). The analysis of the proposed modified VIM is given in Section 3. Then in Section 4, the proposed method is implemented to some examples. Finally, concluding remarks are presented in Section 5.

2. Basic definitions of local fractional calculus:

In this section, we present the basic theory of local fractional calculus Yang (2011,2012).

Definition 1. The LF derivative of $f(x)$ of order α at $x=x_0$ is given by

$$f^{(\alpha)}(x_0)=\lim_{x\to x_0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}} \qquad (2.1)$$

Where $\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(\alpha+1)(f(x)-f(x_0))$.

Definition 2. Let's denote a partition of the interval $[a,b]$ is denoted as (t_j,t_{j+1}), $j=0,...,N-1$, and $t_N=b$ with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_0,\Delta t_1,.....\}$.

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The LF integral of $f(x)$ in the interval $[a,b]$ is given by

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha $$
$$ = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha . \quad (2.2) $$

Definition 3. Let

$$ \frac{1}{\Gamma(1+\alpha)} \int_0^\infty |f(x)|(dx)^\alpha \quad 0 < k < \infty \qquad (2.3) $$

The LF Laplace transform of $f(x)$ is given by

$$ L_\alpha\{f(x)\} = f_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha ,\ 0 < \alpha \le 1 \qquad (2.4) $$

Definition 4. The inverse of the LF Laplace transform of $f(x)$ is

$$ L_\alpha^{-1}\{f_s^{L,\alpha}(s)\} = f(x) = \frac{1}{(2\pi)^\alpha} \int_{\beta-i\omega}^{\beta+i\omega} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(s)(ds)^\alpha \qquad (2.5) $$

where $s^\alpha = \beta^\alpha + i^\alpha \omega^\alpha$, and $\mathrm{Re}(s) = \beta > 0$.

The properties for local fractional Laplace transform used in the paper are given as

$$ L_\alpha\{af(x) + bg(x)\} = af_s^{L,\alpha}(s) + bg_s^{L,\alpha}(s) \qquad (2.6) $$
$$ L_\alpha\{E_\alpha(c^\alpha x^\alpha) f(x)\} = f_s^{L,\alpha}(s-c) \qquad (2.7) $$
$$ L_\alpha\{f^{(k\alpha)}(x)\} = s^{k\alpha} f_s^{L,\alpha}(s) - s^{(k-1)\alpha} f(0) - s^{(k-2)\alpha} f^{(\alpha)}(0) - \cdots - f^{((k-1)\alpha)}(0) \quad (2.8) $$
$$ L_\alpha\{E_\alpha(a^\alpha x^\alpha)\} = \frac{1}{s^\alpha - a^\alpha} \qquad (2.9) $$
$$ L_\alpha\{x^{k\alpha}\} = \frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}} \qquad (2.10) $$

3. Local Fractional Variational Iteration Transform Method (LFVITM).

Let us consider the following local fractional partial differential equations:

$$ L_\alpha u(x,y) + R_\alpha u(x,y) + = g(x,y) ,\ 0 < \alpha \le 1 \qquad (3.1) $$

where $L_\alpha = \frac{\partial^{k\alpha}}{\partial x^{k\alpha}}$, R_α are linear LFDOs and $g(x,y)$ is the source term.

Applying the Yang-Laplace transform (denoted in this paper by $Ł_\alpha$) on both sides of (2.1), we get

Using the property of the Yang-Laplace transform, we have

$$ s^{n\alpha} Ł_\alpha\{u(x,y)\} - s^{(n-1)\alpha} u(0,y) - s^{(n-2)\alpha} u_x^{(\alpha)}(0,y) - \cdots - u_x^{((n-1)\alpha)}(0,y) $$
$$ = Ł_\alpha\{g(x,y) - R_\alpha u(x,y)\}, \quad (3.2) $$

or equivalently

$$ Ł_\alpha\{u(x,y)\} = \frac{1}{s^\alpha} u(0,y) + \frac{1}{s^{2\alpha}} u_x^{(\alpha)}(0,y) + \cdots + \frac{1}{s^{n\alpha}} u_x^{((n-1)\alpha)}(0,y) $$
$$ + \frac{1}{s^{n\alpha}} Ł_\alpha\{g(x,t) - R_\alpha u(x,y)\}. \qquad (3.3) $$

Operating with the Yang-Laplace inverse on both sides of (3.3) gives

$$ u(x,y) = u(0,y) + \frac{x^\alpha}{\Gamma(1+\alpha)} u_x^{(\alpha)}(0,y) + \cdots + \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)} u_x^{((n-1)\alpha)}(0,y) $$
$$ + Ł_\alpha^{-1}\left(\frac{1}{s^{n\alpha}} Ł_\alpha\{g(x,y) - R_\alpha u(x,y)\} \right). \qquad (3.4) $$

Derivative both side (3.4) with respect to x, we have

$$ \frac{\partial^\alpha u(x,y)}{\partial x^\alpha} = \frac{\partial^\alpha}{\partial x^\alpha} Ł_\alpha^{-1}\left(\frac{1}{s^{n\alpha}} Ł_\alpha\{g(x,y) - R_\alpha u(x,y)\} \right) + $$
$$ u_x^{(\alpha)}(0,y) + \cdots + \frac{x^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)} u_x^{((n-1)\alpha)}(0,y). \qquad (3.5) $$

We now structure the correctional local fractional function in the form

$$ u_{m+1}(x,y) = u_m(x,y) + \frac{1}{\Gamma(1+\alpha)} \times $$
$$ \int_0^x \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \left(\begin{array}{l} \frac{\partial^\alpha u_m(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha} Ł_\alpha^{-1}\left(\frac{1}{s^{n\alpha}} Ł_\alpha\{g(\xi,y) - R_\alpha u_m(\xi,y)\} \right) - \\ (u_m)_\xi^{(\alpha)}(0,y) - \cdots - \frac{\xi^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)} (u_m)_\xi^{((n-1)\alpha)}(0,y) \end{array} \right) (d\xi)^\alpha \qquad (3.6) $$

Making the local fractional variation , we get

$$ \delta^\alpha u_{m+1}(x,y) = \delta^\alpha u_m(x,y) + \delta^\alpha \frac{1}{\Gamma(1+\alpha)} \times $$
$$ \int_0^x \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \left(\begin{array}{l} \frac{\partial^\alpha u_m(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha} Ł_\alpha^{-1}\left(\frac{1}{s^{n\alpha}} Ł_\alpha\{g(\xi,y) - R_\alpha u_m(\xi,y)\} \right) - \\ (u_m)_\xi^{(\alpha)}(0,y) - \cdots - \frac{\xi^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)} (u_m)_\xi^{((n-1)\alpha)}(0,y) \end{array} \right) (d\xi)^\alpha $$

The extremum condition of $u_{n+1}(x,y)$ is given by

$$ \delta^\alpha u_{m+1}(x,y) = 0 \qquad (3.7) $$

In view of (3.7), we have the following stationary conditions:

$$ 1 + \left. \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \right|_{\xi=x} = 0, \quad \left. \left(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \right)^{(\alpha)} \right|_{\xi=x} = 0 \quad (3.8) $$

This is turn gives

$$\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)}=-1 \quad (3.9)$$

Substituting (3.9) into (3.6), we obtained

$$u_{m+1}(x,y)=u_m(x,y)-\frac{1}{\Gamma(1+\alpha)}\times \int_0^x\left(\begin{array}{l}\frac{\partial^{\alpha}u_m(\xi,y)}{\partial\xi^{\alpha}}-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}Ł_{\alpha}^{-1}\left(\frac{1}{s^{n\alpha}}Ł_{\alpha}\{g(\xi,y)-R_{\alpha}u_m(\xi,y)\}\right)-\\ (u_m)_{\xi}^{(\alpha)}(0,y)-\cdots-\frac{\xi^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}(u_m)_{\xi}^{((n-1)\alpha)}(0,y)\end{array}\right)(d\xi)^{\alpha} \quad (3.10)$$

Finally, the solution $u(x,y)$ is given by

$$u(x,y)=\lim_{m\to\infty}u_m(x,y) \quad (3.11)$$

4. Illustrate Examples:

Example 1. Let us consider the following Helmholtz equation with local fractional derivative operator in the form

$$\frac{\partial^{2\alpha}H(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}H(x,y)}{\partial y^{2\alpha}}-H(x,y)=0, \quad (4.1)$$

subject to the initial value

$$H(0,y)=0,\ \frac{\partial^{\alpha}H(0,y)}{\partial x^{\alpha}}=\cosh_{\alpha}(y^{\alpha}). \quad (4.2)$$

In view of (3.10) and (4.1) the local fractional iteration algorithm can be written as follows:

$$H_{m+1}(x,y)=H_m(x,y)-\frac{1}{\Gamma(1+\alpha)}\times \int_0^x\left(\frac{\partial^{\alpha}H_m(\xi,y)}{\partial\xi^{\alpha}}-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{H_m(\xi,y)-\frac{\partial^{2\alpha}H_m(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-H_m^{(\alpha)}(0,y)\right)(d\xi)^{\alpha} \quad (4.3)$$

We can use the initial conditions to select $H_0(x,y)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha})$. Using this selection into the correction functional (4.3) gives the following successive approximations

$$H_0(x,y)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha}),$$

$$H_1(x,y)=H_0(x,y)-\frac{1}{\Gamma(1+\alpha)}\times \int_0^x\left(\frac{\partial^{\alpha}H_0(\xi,y)}{\partial\xi^{\alpha}}-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{H_0(\xi,y)-\frac{\partial^{2\alpha}H_0(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-H_0^{(\alpha)}(0,y)\right)(d\xi)^{\alpha}$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha})-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\cosh_{\alpha}(y^{\alpha})-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha})-\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha})\right\}\right)\right)-\cosh_{\alpha}(y^{\alpha})\right)(d\xi)^{\alpha}$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha}),$$

$$H_2(x,y)=H_1(x,y)-\frac{1}{\Gamma(1+\alpha)}\times \int_0^x\left(\frac{\partial^{\alpha}H_1(\xi,y)}{\partial\xi^{\alpha}}-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{H_1(\xi,y)-\frac{\partial^{2\alpha}H_1(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-H_1^{(\alpha)}(0,y)\right)(d\xi)^{\alpha}$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha}),$$

$$\vdots$$

$$H_m(x,y)=H_{m-1}(x,y)-\frac{1}{\Gamma(1+\alpha)}\times \int_0^x\left(\frac{\partial^{\alpha}H_{m-1}(\xi,y)}{\partial\xi^{\alpha}}-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{H_{m-1}(\xi,y)-\frac{\partial^{2\alpha}H_{m-1}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-H_{m-1}^{(\alpha)}(0,y)\right)(d\xi)^{\alpha}$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha}),$$

Finally, the solution of Helmholtz equation (4.1) is given by

$$H(x,y)=\lim_{m\to\infty}H_m(x,y)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha}) \quad (4.4)$$

Example 2. Let us consider the following system of local fractional coupled Helmholtz equations with local fractional derivative:

$$\begin{array}{l}\frac{\partial^{2\alpha}H_1(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}H_2(x,y)}{\partial y^{2\alpha}}-H_1(x,y)=0,\\ \frac{\partial^{2\alpha}H_2(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}H_1(x,y)}{\partial y^{2\alpha}}-H_2(x,y)=0,\end{array} \quad (4.9)$$

subject to the initial conditions

$$\begin{array}{l}H_1(0,y)=0,\ \frac{\partial^{\alpha}H_1(0,y)}{\partial x^{\alpha}}=E_{\alpha}(y^{\alpha}),\\ H_2(0,y)=0,\ \frac{\partial^{\alpha}H_2(0,y)}{\partial x^{\alpha}}=-E_{\alpha}(y^{\alpha}).\end{array} \quad (4.10)$$

Applying local fractional Laplace transform on Eq. (4.9) and using the initial conditions, we have

$$\begin{aligned} & Ł_\alpha\{H_1(x,y)\} = \frac{1}{s^{2\alpha}}E_\alpha(y^\alpha) + \frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_1(x,y) - \frac{\partial^{2\alpha}H_2(x,y)}{\partial y^{2\alpha}}\right\}, \\ & Ł_\alpha\{H_2(x,y)\} = -\frac{1}{s^{2\alpha}}E_\alpha(y^\alpha) + \frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_2(x,y) - \frac{\partial^{2\alpha}H_1(x,y)}{\partial y^{2\alpha}}\right\}. \end{aligned} \tag{4.11}$$

Operating with the local fractional Laplace transform inverse on both sides of Eq. (4.11) we obtain

$$\begin{aligned} & H_1(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha) + Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_1(x,y) - \frac{\partial^{2\alpha}H_2(x,y)}{\partial y^{2\alpha}}\right\}\right), \\ & H_2(x,y) = -\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha) + Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_2(x,y) - \frac{\partial^{2\alpha}H_1(x,y)}{\partial y^{2\alpha}}\right\}\right). \end{aligned} \tag{4.12}$$

Derivative both sides Eq. (4.12) with respect to x, we get

$$\begin{aligned} & \frac{\partial^\alpha H_1(x,y)}{\partial x^\alpha} = E_\alpha(y^\alpha) + \frac{\partial^\alpha}{\partial x^\alpha}Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_1(x,y) - \frac{\partial^{2\alpha}H_2(x,y)}{\partial y^{2\alpha}}\right\}\right), \\ & \frac{\partial^\alpha H_2(x,y)}{\partial x^\alpha} = -E_\alpha(y^\alpha) + \frac{\partial^\alpha}{\partial x^\alpha}Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_2(x,y) - \frac{\partial^{2\alpha}H_1(x,y)}{\partial y^{2\alpha}}\right\}\right). \end{aligned} \tag{4.13}$$

Making the correction function is given

$$\begin{aligned} & H_{1(m+1)}(x,y) = H_{1m}(x,y) - \frac{1}{\Gamma(1+\alpha)}\times \\ & \int_0^x\left(\frac{\partial^\alpha H_{1m}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_{1m}(\xi,y) - \frac{\partial^{2\alpha}H_{2m}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) - E_\alpha(y^\alpha)\right)(d\xi)^\alpha, \\ & H_{2(m+1)}(x,y) = H_{2m}(x,y) - \frac{1}{\Gamma(1+\alpha)}\times \\ & \int_0^x\left(\frac{\partial^\alpha H_{2m}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_{2m}(\xi,y) - \frac{\partial^{2\alpha}H_{1m}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) + E_\alpha(x^\alpha)\right)(d\xi)^\alpha. \end{aligned} \tag{4.14}$$

We can use the initial conditions to select $H_{10}(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha),\ H_{20}(x,y) = -\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha)$. Using this selection into the correction functional (4.14) gives the following successive approximations:

$$\begin{aligned} & H_{10}(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha), \\ & H_{20}(x,y) = -\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha). \end{aligned}$$

$$\begin{aligned} & H_{11}(x,y) = H_{10}(x,y) - \frac{1}{\Gamma(1+\alpha)}\times \\ & \int_0^x\left(\frac{\partial^\alpha H_{10}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_{10}(\xi,y) - \frac{\partial^{2\alpha}H_{20}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) - E_\alpha(y^\alpha)\right)(d\xi)^\alpha \\ & H_{21}(x,y) = H_{20}(x,y) - \frac{1}{\Gamma(1+\alpha)}\times \\ & \int_0^x\left(\frac{\partial^\alpha H_{20}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_{20}(\xi,y) - \frac{\partial^{2\alpha}H_{10}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) + E_\alpha(x^\alpha)\right)(d\xi)^\alpha \end{aligned}$$

$$\begin{aligned} & = E_\alpha(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{2x^{3\alpha}}{\Gamma(1+3\alpha)}\right), \\ & = -E_\alpha(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{2x^{3\alpha}}{\Gamma(1+3\alpha)}\right), \end{aligned}$$

$$\begin{aligned} & H_{12}(x,y) = H_{11}(x,y) - \frac{1}{\Gamma(1+\alpha)}\times \\ & \int_0^x\left(\frac{\partial^\alpha H_{11}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_{11}(\xi,y) - \frac{\partial^{2\alpha}H_{21}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) - E_\alpha(y^\alpha)\right)(d\xi)^\alpha \\ & H_{22}(x,y) = H_{21}(x,y) - \frac{1}{\Gamma(1+\alpha)}\times \\ & \int_0^x\left(\frac{\partial^\alpha H_{21}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_{21}(\xi,y) - \frac{\partial^{2\alpha}H_{11}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) + E_\alpha(x^\alpha)\right)(d\xi)^\alpha \end{aligned}$$

$$\begin{aligned} & = E_\alpha(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{2x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{4x^{5\alpha}}{\Gamma(1+5\alpha)}\right), \\ & = -E_\alpha(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{2x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{4x^{5\alpha}}{\Gamma(1+5\alpha)}\right). \end{aligned}$$

$$\vdots$$

$$\begin{aligned} & H_{1m}(x,y) = E_\alpha(y^\alpha)\sum_{k=0}^m \frac{2^k x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}, \\ & H_{2m}(x,t) = -E_\alpha(y^\alpha)\sum_{k=0}^m \frac{2^k x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}. \end{aligned}$$

Therefore, the series solutions can be written in the form

$$\begin{aligned} & H_1(x,y) = \lim_{m\to\infty} H_{1m}(x,y) = E_\alpha(y^\alpha)\frac{\sinh_\alpha\left(\sqrt{2}x^\alpha\right)}{\sqrt{2}}, \\ & H_2(x,y) = \lim_{m\to\infty} H_{2m}(x,y) = -E_\alpha(y^\alpha)\frac{\sinh_\alpha\left(\sqrt{2}x^\alpha\right)}{\sqrt{2}}. \end{aligned} \tag{4.15}$$

5. Conclusions:

In this work we considered the coupling method of the local fractional variational iteration method and Laplace transform to solve Helmholtz and coupled Helmholtz equations and their approximate solutions were obtained. The results include an efficient implement of the local fractional variational iteration transform method to solve the partial differential equations with local fractional derivative operator.

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