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Bifurcation of Solution in Singularly Perturbed ODEs by Using Lyapunov Schmidt Reduction

*A.H.Kamil K. H. Yasir

Department of Mathematics- University of Thi-Qar

***Email:** aahhkk1992@gmail.com

Email: istathj@yahoo.com

Abstract:

This paper aims to study the bifurcation of solution in singularly perturbed ODEs:

$$\dot{x}= F(x, y, \epsilon),$$
$$\epsilon\dot{y}=G(x,y, \epsilon),$$

the hypothesis

$$rank D_y g(x_0, y_0, \epsilon) = m - 1$$

the bifurcation of solution in the ODE system will be studied by effect of the system by using Lyapunov Schmidt reduction. Is the study of behaviour of solution of singularly perturbed ODEs when perturbation parameter $0 < \epsilon \ll 1$. The bifurcation of solution in this kind of ordinary differential equation was studied in n-dimensional. Sufficient conditions for the system to undergoes (fold,transcritical and pitchfork) bifurcation are given. The ODE will be reduced to an equivalent system by using Lyapunov Schmidt reduction method. Moreover, for this purpose of obtaining curve of the system (Fast-Slow system).

Keywords: ODEs, Bifurcation, Singularly perturbed ODEs, Lyapunov Schmidt Re- duction.

1.Introduction:

It is known that many of nonlinear problems that appear in mathematics and physics can be written in the form of operator equation,

$$F(x, y, \epsilon) = 0, \quad x \in R^n, y \in R^m, \epsilon \in R. \qquad (1.1)$$

Where F is smooth map. For these problems, the method of the reduction to finite dimen- sional equation.

$$\acute{x} = F(x, y, \epsilon), \quad (1.2)$$

when $0 < \epsilon \ll 1$ $\quad \epsilon\acute{y}= G(x, y, \epsilon), \quad (1.3)$

The method of finite dimensional reduction was introduced by [Lyapunov (1906)] and [Schmidt (1908)]. They have introduced this method to find the solution of equations similar to the equations (1.1). [Vainberg(1969)], [Loginov(1985)] and [Sapronov (1973,1991)] transform equation (1.1) into (1.2) by using Lyapunov Schmidt Reduction with the condition that, equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, etc). In the method of finite dimen- sional reduction the solutions of equations of finite dimensional spaces coincide with the solutions of the equations of finite dimensional spaces [Sapronov (2004)]. [Yasir(2007)] investigate the asymptotic stability of an equilibrium solution of the differential algebraic equations (DAEs) by reducing such DAEs by Liapunov Schmidt reduction to a corresponding one. [Shanan(2013)] used the method of Lyapunov Schmidt to study the bifurcation solutions of system of nonlinear differential equations.

Consider the fast ODEs:

$$\acute{x} = f(x, y, \epsilon), \qquad (1.4)$$

$$\epsilon \acute{y} = g(x, y, \epsilon), \quad (1.5)$$

where $(f, g) : R^n \times R^m \times R \to R^n \times R^m$, $0 < \epsilon \ll 1$.
Define the following related sets:

$$M = \{(x, y, \epsilon) \in R^n \times R^m \times R : f(x, y, \epsilon) = g(x, y, \epsilon) = 0\}, \quad (1.6)$$

and the set:

$$T = M \setminus S, \quad (1.7)$$

where S is defined by:

$$S = \{(x, y, \epsilon) \in M : rankD_y g(x, y, \epsilon) = m - 1\}. \quad (1.8)$$

Let $(x_0, y_0, \epsilon) \in M$ such that $f(x_0, y_0, \epsilon) = 0$. If $rankD_y g(x_0, y_0, \epsilon) = m$ then $(x_0, y_0, \epsilon) \in T$
and it is just a non-degenerate equilibrium point, the rank condition

$$rankD_y g(x, y, \epsilon) = m - 1.$$

Since $D_y g(x, y, \epsilon)$ is singular at singular point (x_0, y_0, ϵ), the solution may bifurcate at that point, there may be impasse for which the solution does not exist near that point, or the solution is well defined through the singularly. Our study includes the stability of degenerate equilibrium points $(x_0, y_0, \epsilon) \in S$ of the ODEs for which the solution near that point exists and well is defined. Let $(x_0, y_0, \epsilon) \in M$ be an equilibrium point for i.e. $f(x_0, y_0, \epsilon) = 0$ and

$$rankD_y g(x_0, y_0, \epsilon) = m - 1. \quad (1.9)$$

The assumption (1.9) states that zero is an eigenvalue of $D_y g(x_0, y_0, \epsilon)$.

2.Basic ideas:

Definition 2.1 *[Kuehn(2015, Fenichel (1979)] ((m, n)-Fast-Slow System):*
System of ordinary differential equations has the form:

$$\frac{dx}{d\tau} = \acute{x} = \epsilon f(x, y, \epsilon) \quad (2.1)$$
$$\frac{dy}{d\tau} = \acute{y} = g(x, y, \epsilon) \quad (2.2)$$

is called a m fast-slow system.
where variable x is called fast variable, variable y is called slow variable. A time-scale decomposition of the singularly perturbed system yields reduced –order representations for the slow and fast subsystems. More specifically in the limit $\epsilon \to 0$ the fast dynamics become instantaneous in the slow time-scale t. By applying the time scale:

setting $t = \tau\epsilon \Rightarrow \tau = \frac{t}{\epsilon}$

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} \Rightarrow \frac{dx}{dt} = \frac{dx}{d\tau}\frac{1}{\epsilon} = f(x, y, \epsilon)$$

and

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{\epsilon}\frac{dy}{d\tau} = \frac{1}{\epsilon}g(x, y, \epsilon) \Rightarrow \epsilon\frac{dy}{dt} = g(x, y, \epsilon)$$

that gives the equivalent form:

$$\frac{dx}{dt} = \dot{x} = f(x, y, \epsilon), \quad (2.3)$$
$$\frac{dy}{dt} = \epsilon\dot{y} = g(x, y, \epsilon), \quad (2.4)$$

the systems (2.3), (2.4) is called n fast-slow system. It refers to t as the fast time scale or fast time and to τ as the slow time scale or slow time.
When ϵ approaches to 0 for system (2.3), (2.4) we get:

$$\dot{x} = f(x, y, 0),$$
$$0 = g(x, y, 0),$$

which represent to DAEs with index one, and it can be readily reduced to an ODEs. Sometimes, one finds that the x variables are slow and the y variables are fast with similar or no changes regarding the notation for the functions f and g.

Definition 2.2*[OMalley(1991)]*
The DAEs that obtained by setting ϵ approaches to zero in the formulation of the slow time scale (2.1), (2.2) is called the slow subsystem or slow vector field:

$$0 = f(x, y, 0), \quad (2.5)$$
$$\dot{y} = g(x, y, 0). \quad (2.6)$$

The flow generated by (2.5), (2.6) is called the slow flow.
The slow subsystem is also referred to as the reduced problem and its flow as the reduced flow.

Definition 2.3 *[De-Jager(1996)]:*
The singularly perturbed ODEs obtained by setting ϵ approaches to 0
on the fast time scale formulation (2.3), (2.4) is called a fast subsystem or fast vector field:

$$\dot{x} = f(x, y, 0), \quad (2.7)$$
$$0 = g(x, y, 0). \quad (2.8)$$

The flow of (2.7), (2.8) is called the fast flow.

3.Bifurcation:

Bifurcation is a French word that has been introduced into nonlinear dynamics by (Poincare et al.

1899). Bifurcation theory studies the change in behavior of the system with the change in parameters that involves the change in the dynamics behavior. These changes are only qualitative in nature. But there may be changes in situations as well. In bifurcation problems, it is useful to consider a space formed by using the state variables and the control parameters, called the state-control space. The definition of bifurcation is as follows:

Definition 3.1 *[Kuznetsov(1998)]:*

The appearance of a topologically nonequivalent phase portrait under variation of parameters is called **a bifurcation**.
Thus, a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value.

Definition 3.2*(Bifurcation point[Sastry(1999)]:*

The sudden change in the behavior of the system when a parameter passes through a critical value.

Definition 3.3*(Bifurcation diagram [Kuznetsov (1998)]):*

A bifurcation diagram of the dynamical system is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each stratum.

Thus, bifurcation is a complex phenomena occurs in nonlinear systems, it is refers to the branching of solutions at some critical value parameters, which results in a loss of the structural stability and it is one of routes to chaos[Panfilov, Maree(2005)]. Here we will state the bifurcation kinds such as (fold, transcritical and pitchfork bifurcation).

4. Singularly Perturbed ODEs [OMalley (1991)]:

Singular perturbation problem is a differential equation with another condition having a small parameter that is multiplying the highest derivatives, it is regard one of the important sources for DAE problems. The fundamental reason for studying singularly perturbed theory is to consider a problem with a small parameter s and state a solution $x(t,\epsilon)$. Also defined is an unpertinurbed (neighbouring) problem with solution $x(t, 0)$. If $\lim_{t\to 0}$ "$x(t,\epsilon) - x(t, 0)$" does not lead to zero when ϵ approaches to zero. A singular perturbation occurs whenever the limit of regular perturbation problem fails. Here we look at systems of the standard form:

$$\dot{x} = f(x, y, t, \epsilon), \quad (4.1)$$
$$\epsilon\, \dot{y} = g(x, y, t, \epsilon), \quad (4.2)$$
$$x(0) = x_0,\ y(0) = y_0. \quad (4.3)$$

Where $f : R^n \times R^m \times R \times R \to R^n$, $g : R^n \times R^m \times R \times R \to R^n$, $x \in R^n$, $y \in R^m$, ϵ is a perturbation parameter.

5. Liapunov-Schmidt reduction in ODEs[Yasir(2007)]:

In this section we investigate the bifurcation of an equilibrium solution of the ordinary differential equations (ODEs)
Consider the ODEs:

$$\epsilon\, \acute{x} = f(x, y, \epsilon), \quad (5.1)$$
$$\acute{y} = g(x, y, \epsilon), \quad (5.2)$$

and assume that the equilibrium point is (0, 0) for $0 < \epsilon \ll 1$ such that the conditions where $(f, g) : R^n \times R^m \times R \to R^n \times R^m$ are C^1. and the rank condition defined as:

$$rank D_y g(x_0, y_0, \epsilon) = m - 1, \quad (5.3)$$

are satisfied. Let $D_y g(0, 0) = B$ then from conditions (5.3) we have $rank(B)(0, 0, 0) = m - 1$. Choose complements vector spaces H and N to $kerB$ and $rangeB$ respectively. Then

$$R^m = kerB \oplus H, \quad (5.4)$$
$$R^m = N \oplus rangeB. \quad (5.5)$$

Then we conclude that $dimH = m - 1$ and $dimN = 1$. Define the projections$E : R^m \to rangeB$ and the complementary projection $(I - E) : R^m \to N$ such that the ODEs (4.1),(4.2) expanded to an equivalent pairs of equations

$$\acute{x} = f(x, y, \epsilon), \quad (5.6)$$
$$\acute{y} = \epsilon\, Eg(x, y, \epsilon), \quad (5.7)$$

and

$$\acute{x} = f(x, y, \epsilon), \quad (5.8)$$
$$\acute{y} = \epsilon\, (I - E)g(x, y, \epsilon). \quad (5.9)$$

Because of this splitting any vector $y \in R^m$ can be decomposed in the form $y = v + w$, where $v \in kerB$ and $w \in H$. Then the equation (5.6),(5.7) can be written as:

$$\dot{x} = f(x, v + w, \epsilon), \quad (5.10)$$
$$\dot{y} = \epsilon\, Eg(x, v + w, \epsilon). \quad (5.11)$$

Then in (5.11) the second equation can be considered as a map $\varphi : R^n \times kerB \times H \times R \to rangeB$, where

$$\varphi(x, v, w, \epsilon) = \epsilon\, Eg(x, v + w, \epsilon).$$

Now we have

$$\left(\frac{\partial\, \epsilon\, Eg(x, v + w, \epsilon)}{\partial w}\right)(0, 0, 0) = \epsilon\, EB.$$

Since E act as the identity map on $rangeB$ so

$$\left(\frac{\partial\, \epsilon\, Eg(x, v + w, \epsilon)}{\partial w}\right)(0, 0, 0) = \epsilon\, B,$$

and since $B : H \to rangeB$, has a full rank at $(0, 0, \epsilon)$, it follows from the implicit function theorem that the second equation of (5.7) can be solved uniquely for w near $(0, 0, \epsilon)$. i.e., $w = W(x, v, \epsilon)$, where $W : R^n \times kerB \times R \to M$ satisfies:

$$\epsilon\, Eg(x, v + W(x, v, \epsilon), \epsilon) \equiv 0,$$
$$W(0, 0, \epsilon) = 0. \quad (5.12)$$

From (5.11) and from ODEs (4.1),(4.2) we get the reduced ODEs:

$$\dot{x} = F(x, v, \epsilon), \quad (5.13)$$
$$\dot{y} = \epsilon\, G(x, v, \epsilon), \quad (5.14)$$

where $(F, G) : R^n \times kerB \times R \to R^n \times N$ defined by:

$$F(x, y, \epsilon) = f(x, v + W(x, v, \epsilon), \epsilon), \quad (5.15)$$
$$G(x, y, \epsilon) = \epsilon\, (I - E)g(x, v + W(x, v, \epsilon), \epsilon). \quad (5.16)$$

Now the Lyapunov Schmidt reduction will generalized to n−dimensional subspace when perturbed parameter $0 < \epsilon \ll 1$.

$$\dot{x} = F(x, v, \epsilon), \quad (5.17)$$
$$\epsilon\, \dot{y} = G(x, v, \epsilon), \quad (5.18)$$

Where

$$(F, G) : R^n \times kerB \times R \to R^n \times N,$$

defined by

$$F(x, v, \epsilon) = f(x, v + W(x, v, \epsilon), \epsilon), \quad (5.19)$$
$$G(x, v, \epsilon) = \epsilon\, (I - E)g(x, v + W(x, v, \epsilon), \epsilon). \quad (5.20)$$

To see this choose explicit coordinate on $kerB$ and N. For this purpose assume v and $v_0{}^*$ be none-zero vectors in $kerB$ and $(rangeB)^{\perp}$ respectively. Then the vector $v \in kerB$ can be uniquely written in the form $v = yv_0$ where $y \in R$.

Define

$$\tilde{G}(x, y, \epsilon) = < v_0{}^*, G(x, yv_0, \epsilon) >,$$

where G is reduced equation (5.17). Now we show that $\tilde{G}(x, y, \epsilon) = 0$ iff $G(x, yv_0, \epsilon) = 0$ so the zeros of $\tilde{G}$ are one to one correspondence with the solutions of $g(x, y, \epsilon) = 0$. Then the function $\tilde{G}$ can be written in terms of the original ODEs (1.2), (1.3) by using (5.19), (5.20) (i.e)

$$\tilde{G}(x, y, \epsilon) = < v_0{}^*, \epsilon\, g(x, yv_0 + W(x, yv_0, \epsilon), \epsilon) > . \quad (5.21)$$

The function $\tilde{G}$ is the reduced function to the constraint equation g in the ODEs (1.2), (1.3) in a new change of coordinates. Also the relation between $\tilde{G}$ and G is that $\tilde{G}$ is just a representation of G in new coordinates. Hence the reduced ODEs in new coordinate are given by

$$\dot{x} = \tilde{F}(x, y, \epsilon), \quad (5.2)$$
$$\dot{y} = \tilde{G}(x, y, \epsilon), \quad (5.23)$$

where $\tilde{F}, \tilde{G} : R \times R^n \times R \to R^n$ such that $\tilde{F}$ defined by

$$\tilde{F}(x, y, \epsilon) = f(x, yv_0 + W(x, yv_0, \epsilon), \epsilon), \quad (5.24)$$

and $\tilde{G}$ as defined in (5.21). As we mentioned above $\tilde{G}(x, y, \epsilon) = 0$ iff $G(x, yv_0, \epsilon) = (I - E)g(x, yv_0 + W(x, yv_0, \epsilon), \epsilon) = 0$. Thus we have:

$$\frac{\partial G}{\partial y}(x, yv0, \epsilon) = \epsilon(I - E)\frac{\partial g}{\partial y}(x, yv_0 + W(x, yv_0, \epsilon), \epsilon)\left(v_0 + \frac{\partial W}{\partial y}\right)$$

On evaluating at $(0, 0, \epsilon)$ we have

$$\frac{\partial G}{\partial y}(0,0,\epsilon) = \epsilon(I - E)B\left(v_0 + \frac{\partial W}{\partial y}(0,0,\epsilon)\right).$$

Since $(I - E)B = 0$, so $\frac{\partial G}{\partial y}(0,0,\epsilon)$. By a similar way we get $\frac{\partial \tilde{G}}{\partial y}(0,0,\epsilon) = 0$.

That means the reduced ODEs have a singularly at $(0, 0, \epsilon)$.

5.1 Fold bifurcation in R^n**:**

A fold bifurcation point is a pair of equilibria, meets and disappears with a zero eigenvalue [Perko(2001)]. One of the equilibria (saddle) is unstable while the other (node) is stable [Sastry(1999)]. Now, consider the ODEs (5.1),(5.2). We will study fold bifurcation of the singularly parameterized ODEs system by the following theorem:

Theorem 5.1 Consider the ODEs (5.1),(5.2) defined.on $S_0 = \{x^*, y^*, \epsilon^* \in R^n \times R^m \times R : f(x^*, y^*, \epsilon^*) = g(x^*, y^*, \epsilon^*) = 0\}$ with an equilibrium point $(0, 0, \epsilon)$

and the non-hyperbolic conditions $\frac{\partial f}{\partial x}(0,0,\epsilon)=0$, $\frac{\partial g}{\partial x}(0,0,\epsilon)=0, \frac{\partial^2 f}{\partial x\partial y}(0,0,\epsilon)=0$. If the following conditions are hold

1. $< v_0{}^*, \frac{\partial g}{\partial \epsilon}(0,0,\epsilon) > \neq 0$,
2. $< v_0{}^*, \frac{\partial^2 g}{\partial x^2}(0,0,\epsilon)(v,v) > \neq 0$.

Then (0, 0, ϵ) is a fold bifurcation point for the reduced ODEs (5.13),(5.14), Lyapunov Schmidt reduction is locally equivalent to one of the following normal forms:

$$\frac{d\eta}{dt} = \pm\mu \pm \eta^2$$

Proof: Suppose that $(x, y, \epsilon) = (0, 0, \epsilon)$ is critical point and consider the reduced ODEs (5.13),(5.17) obtained by Lyapunov Schmidt reduction. Differentiate the bifurcation equation (5.17) w.r.t.α we get:

$$\frac{\partial G}{\partial \in}(x^*,y^*,\in^*)=\in\left[(I-E)\left(\begin{matrix}\frac{\partial g}{\partial x}(x^*,y^*,\in^*)\frac{\partial x}{\partial \in}(x^*,y^*,\in^*)+\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\\ \left[\frac{\partial W}{\partial \in}(x^*,y^*,\in^*)\right]+(I-E)g(x^*,y^*,\in^*)\end{matrix}\right)\right]+\left(\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\right)$$

and

$$\frac{\partial G}{\partial \in}(x^*,y^*,\in^*)=\in\left[(I-E)\left(\begin{matrix}\frac{\partial g}{\partial x}(x^*,y^*,\in^*)\frac{\partial x}{\partial \in}(x^*,y^*,\in^*)+\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\\ \left[\frac{\partial W}{\partial \in}(x^*,y^*,\in^*)\right]+\left(\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\right)\end{matrix}\right)\right]+(I-E)g(x^*,y^*,\in^*)$$

Evaluate at $(0, 0, \epsilon)$, we get:

$$\frac{\partial G}{\partial \in}(x^*,y^*,\in^*)=\in\left[(I-E)\left(\begin{matrix}\frac{\partial g}{\partial x}(0,0,\in)\frac{\partial x}{\partial \in}(0,0,\in)+\frac{\partial g}{\partial y}(0,0,\in)\\ \left[\frac{\partial W}{\partial \in}(0,0,\in)\right]+\left(\frac{\partial g}{\partial y}(0,0,\in)\right)\end{matrix}\right)\right]+(I-E)g(0,0,\in)$$

from rank condition (5.3) we get:

$$\frac{\partial G}{\partial \epsilon}(x^*,y^*,\epsilon^*)=\epsilon\left[(I-E)\left(\frac{\partial g}{\partial \epsilon}(0,0,\epsilon)\right)\right]+(I-E)g(0,0,\epsilon),$$

Then from condition (1) we get:

$$\frac{\partial G}{\partial \epsilon}(x^*,y^*,\epsilon^*)\neq 0$$

To prove the second condition we differentiate bifurcation equation (5.14) w.r.t.x twice:

$$\frac{\partial^2 G}{\partial x^2}(x^*,y^*,\in^*)=\frac{\partial}{\partial x}\left[\in(I-E)\left(\begin{matrix}\frac{\partial g}{\partial x}(x^*,y^*,\in^*)+\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\\ \frac{\partial y}{\partial x}(x^*,y^*,\in^*)+\frac{\partial g}{\partial x}(0,0,\in^*)\frac{\partial \in}{\partial x}(x^*,y^*,\in^*)\end{matrix}\right)\right]$$

$$+(I-E)\left[\frac{\partial \in}{\partial x}(x^*,y^*,\in^*)g(x^*,y^*,\in^*)\right]$$

$$\frac{\partial^2 G}{\partial x^2}(x^*,y^*,\in^*)=\left[\in(I-E)\left(\begin{matrix}\frac{\partial^2 g}{\partial x^2}(x^*,y^*,\in^*)+\frac{\partial^2 g}{\partial x\partial y}(x^*,y^*,\in^*)\\ \frac{\partial W}{\partial x}(x^*,y^*,\in^*)+\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\frac{\partial^2 W}{\partial x^2}(x^*,y^*,\in^*)\end{matrix}\right)\right]$$

Evaluate at $(0,0,\epsilon)$ we get:

$$\frac{\partial^2 G}{\partial x^2}(0,0,\in)=\left[\in(I-E)\left(\begin{matrix}\frac{\partial^2 g}{\partial x^2}(0,0,\in)+\frac{\partial^2 g}{\partial x\partial y}(0,0,\in)\\ \frac{\partial W}{\partial x}(0,0,\in)+\frac{\partial g}{\partial y}(0,0,\in)\frac{\partial^2 W}{\partial x^2}(0,0,\in)\end{matrix}\right)\right]$$

Then from rank condition (5.3), and from the condition above given in theorem we get:

$$\frac{\partial^2 G}{\partial x^2}(0,0,\epsilon)=\epsilon(I-E)\left[\frac{\partial^2 g}{\partial x^2}(0,0,,\epsilon)\right]$$

and from condition 2 we see that:

$$\frac{\partial^2 G}{\partial x^2}(0,0,\epsilon)\neq 0$$

So the bifurcation equation (5.14) satisfy fold bifurcation conditions and it is locally equivalent to one of the following normal forms $\frac{d\eta}{dt} = \pm\mu \pm \eta^2$

5.2 Pitchfork bifurcation in R^n:

In the pitchfork bifurcation, an equilibrium point reverses its stability, and two new equi- librium points are born [Perko(2001)]. Now we will state the pitchfork bifurcation theorem for the singularly parameterized ODEs(5.1),(5.2) as follows:

Theorem 5.2 Consider the ODEs (5.1),(5.2), defined on S_0 with an equilibrium point(0, 0,ϵ). and suppose that the non-hyperbolic conditions

$\frac{\partial f}{\partial x}(0,0,\epsilon)=0, \frac{\partial g}{\partial x}(0,0,\epsilon)=0, \frac{\partial^2 f}{\partial x\partial y}(0,0,\epsilon)=0$

are satisfied. If the following condition are hold:

1. $< v_0{}^*, \frac{\partial g}{\partial \epsilon}(0,0,\epsilon) >= 0$,
2. $< v_0{}^*, \frac{\partial^2 g}{\partial x^2}(0,0,\epsilon)(v,v) >= 0$,

3. $<\upsilon_0^*, \frac{\partial^3 g}{\partial x^3}(0,0,\in)(\upsilon,\upsilon,\upsilon)>\neq 0, <\upsilon_0^*,$

$$\frac{\partial^2 g}{\partial x \partial \in}(0,0,\in)(\upsilon)>\neq 0$$

Then (0, 0, ϵ) is a pitchfork bifurcation point for the reduced ODEs (5.13),(5.14), when $0 < \epsilon \ll 1$.

Proof : Suppose that (x^*, y^*, ϵ *) = (0, 0, ,ϵ) is critical point and we differentiate (5.14) w.r.t. ϵ as in theorem (5.1) we get:

$$\frac{\partial G}{\partial \in}(0,0,\in)=\in\left[(I-E)\left(\frac{\partial g}{\partial \in}(0,0,\in)\right)\right]+(I-E)g(0,0,\in)$$

Then from condition (1) we get:

$$\frac{\partial G}{\partial \epsilon}(0{,}0,\epsilon) = 0.$$

To prove the second condition we differentiate bifurcation equation (5.14) w.r.t.x twice as in theorem(5.1) we get:

To prove the second condition we differentiate bifurcation equation (5.14) w.r.t.x twice as in theorem (5.1) we get:

$$\frac{\partial^2 G}{\partial x^2}(0,0,\in)=\left[\in(I-E)\begin{pmatrix}\frac{\partial^2 g}{\partial x^2}(0,0,\in)+\frac{\partial^2 g}{\partial x \partial y}(0,0,\in)\\ \frac{\partial W}{\partial x}(0,0,\in)+\frac{\partial g}{\partial y}(0,0,\in)\frac{\partial^2 W}{\partial x^2}(0,0,\in)\end{pmatrix}\right]$$

and from condition 2 we see that

$$\frac{\partial^2 G}{\partial x^2}(0{,}0,\epsilon) = \epsilon(I-E)\left[\frac{\partial^2 g}{\partial x^2}(0{,}0,,\epsilon)\right]$$

then

$$\frac{\partial^2 G}{\partial x^2}(0{,}0,\epsilon) = 0$$

To prove the third condition we differentiate bifurcation equation (5.14) w.r.t.x and ϵ we get:

$$\frac{\partial^2 G}{\partial x \partial \in}\left(x^*,y^*,\in^*\right)=(I-E)\left(\frac{\partial}{\partial x}\right)(\in)\begin{pmatrix}\frac{\partial g}{\partial x}(x^*,y^*,\in^*)+\frac{\partial x}{\partial \in}(x^*,y^*,\in^*)+\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\\ \left[\frac{\partial W}{\partial \in}(x^*,y^*,\in^*)\right]+\frac{\partial g}{\partial \in}(x^*,y^*,\in^*)\end{pmatrix}(I-E)g\left(x^*,y^*,\in^*\right)$$

and

$$\frac{\partial^2 G}{\partial x \partial \in}\left(x^*,y^*,\in^*\right)=(I-E)\in\begin{pmatrix}\frac{\partial^2 g}{\partial x^2}(x^*,y^*,\in^*)\frac{\partial x}{\partial \in}(x^*,y^*,\in^*)+\frac{\partial g}{\partial x}(x^*,y^*,\in^*)\frac{\partial^2 x}{\partial \in^2}(x^*,y^*,\in^*)+\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\\ \left[\frac{\partial^2 W}{\partial x \partial \in}(x^*,y^*,\in^*)\right]+\frac{\partial^2 g}{\partial x \partial y}(x^*,y^*,\in^*)\left[\frac{\partial W}{\partial \in}(x^*,y^*,\in^*)\right]+\frac{\partial^2 g}{\partial x \partial \in}(x^*,y^*,\in^*)\end{pmatrix}$$

Evaluate at $(0, 0, \epsilon)$

$$\frac{\partial^2 G}{\partial x \partial \in}(0y0,\in)=(I-E)\in\begin{pmatrix}\frac{\partial^2 g}{\partial x^2}(0y0,\in)\frac{\partial x}{\partial \in}(0y0,\in)+\frac{\partial g}{\partial x}(0y0,\in)\frac{\partial^2 x}{\partial \in^2}(0y0,\in)+\frac{\partial g}{\partial y}(0y0,\in)\\ \left[\frac{\partial^2 W}{\partial x \partial \in}(0y0,\in)\right]+\frac{\partial^2 g}{\partial x \partial y}(0y0,\in)\left[\frac{\partial W}{\partial \in}(0y0,\in)\right]+\frac{\partial^2 g}{\partial x \partial \in}(0y0,\in)\end{pmatrix}$$

Then from rank condition (5.3), and condition above given in theorem we get:

$$\frac{\partial^2 G}{\partial x \partial \epsilon}(0,0,\epsilon) = (I-E)\left[\epsilon\left(\frac{\partial^2 g}{\partial x \partial \epsilon}(0,0,\epsilon)\right)\right],$$

and from condition 3 we see that:

$$\frac{\partial^2 G}{\partial x \partial \epsilon}(0,0,\epsilon) \neq 0$$

To prove $\frac{\partial^3 G}{\partial x^3}(x^*,y^*,\epsilon^*)$, we differentiate (5.14) w.r.t.x three times as follows:

$$\frac{\partial^3 G}{\partial x^3}\left(x^*,y^*,\in^*\right)=\left(\frac{\partial}{\partial x}\right)\left(\in(I-E)\begin{pmatrix}\frac{\partial^2 g}{\partial x^2}(x^*,y^*,\in^*)+\frac{\partial^2 g}{\partial x \partial y}(x^*,y^*,\in^*)\\ \frac{\partial W}{\partial x}(x^*,y^*,\in^*)+\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\frac{\partial^2 W}{\partial x^2}(x^*,y^*,\in^*)\end{pmatrix}\right)$$

and

$$\frac{\partial^3 G}{\partial x^3}\left(x^*,y^*,\in^*\right)=\in(I-E)\begin{pmatrix}\frac{\partial^3 g}{\partial x^3}(x^*,y^*,\in^*)+\frac{\partial^3 g}{\partial x^2 \partial y}(x^*,y^*,\in^*)\frac{\partial W}{\partial x}(x^*,y^*,\in^*)+\\ \frac{\partial^2 g}{\partial x \partial y}(x^*,y^*,\in^*)\frac{\partial^2 W}{\partial x^2}(x^*,y^*,\in^*)+\frac{\partial^2 g}{\partial x \partial y}(x^*,y^*,\in^*)\\ \frac{\partial^2 W}{\partial x^2}(x^*,y^*,\in^*)+\frac{\partial g}{\partial y}(x^*,y^*,\in^*)\frac{\partial^3 W}{\partial x^3}(x^*,y^*,\in^*)\end{pmatrix}$$

Evaluate at (0, 0, ϵ),and condition above given in theorem we get:

$$\frac{\partial^3 G}{\partial x^3}(0,0,\in) = \in (I-E)\begin{pmatrix} \frac{\partial^3 g}{\partial x^3}(0,0,\in)+\frac{\partial^3 g}{\partial x^2\partial y}(0,0,\in)\frac{\partial W}{\partial x}(0,0,\in)+ \\ \frac{\partial^2 g}{\partial x\partial y}(0,0,\in)\frac{\partial^2 W}{\partial x^2}(0,0,\in)+\frac{\partial^2 g}{\partial x\partial y}(0,0,\in) \\ \frac{\partial^2 W}{\partial x^2}(0,0,\in)+\frac{\partial g}{\partial y}(0,0,\in)\frac{\partial^3 W}{\partial x^3}(0,0,\in) \end{pmatrix}$$

Then from rank condition (5.3), and condition above given in theorem we get :

$$\frac{\partial^3 G}{\partial x^3}(0,0,\epsilon) = \epsilon(I-E)\left[\frac{\partial^3 g}{\partial x^3}(0,0,\epsilon)\right],$$

and from condition 3 we see that:

$$\frac{\partial^3 G}{\partial x^3}(0,0,\epsilon) \neq 0$$

So the bifurcation equation (5.14) satisfy pitchfork bifurcation conditions, and it is locally equivalent to one of the following normal forms:

$$\frac{d\eta}{dt} = \pm\mu\eta \pm \eta^3$$

5.3 Transcritical bifurcation in R^n:

A transcritical bifurcation is one in which an equilibrium point exists for all values of a parameter and is never destroyed [Perko(2001)]. In transcritical bifurcation there is an exchange of stability between two equilibrium points, there is one unstable and the other is stable equilibrium point. Now we will introduce the transcritical bifurcation theorem for the singularly parameterized ODEs as follows:

Theorem 5.3 Consider the ODEs (5.1),(5.2), defined on S_0 with an equilibrium point (0, 0,ϵ).and suppose that the non-hyperbolic conditions

$\frac{\partial f}{\partial x}(0,0,\epsilon)=0, \frac{\partial g}{\partial x}(0,0,\epsilon)=0, \frac{\partial^2 f}{\partial x\partial y}(0,0,\epsilon)=0$

are satisfied. If the following condition are hold:

$$4. < {v_0}^*, \frac{\partial g}{\partial \epsilon}(0,0,\epsilon) >= 0,$$

$$5. < {v_0}^*, \frac{\partial^2 g}{\partial x^2}(0,0,\epsilon)(v,v) >\neq 0,$$

$$6. < {v_0}^*, \frac{\partial^2 g}{\partial x\partial \epsilon}(0,0,\epsilon)(v) >\neq 0.$$

Then (0, 0, ϵ) is a transcritical bifurcation point for the singularly parameterized ODEs (5.13),(5.14), when $0 < \epsilon \ll 1$.

Proof : Suppose that $(x^*, y^*, \epsilon^*) = (0, 0, \epsilon)$ is critical point, we differentiate (5.14) w.r.t. ϵ as in theorem(5.1) we get:

$$\frac{\partial G}{\partial \epsilon}(x^*,y^*,\epsilon^*) = \epsilon\left[(I-E)\left(\frac{\partial g}{\partial \epsilon}(0,0,\epsilon)\right)\right] + (I-E)g(0,0,\epsilon)$$

Then from condition (1) we get:

$$\frac{\partial G}{\partial \epsilon}(x^*,y^*,\epsilon^*) = 0,$$

and

$$\frac{\partial^2 G}{\partial x^2}(0,0,\epsilon) = \epsilon(I-E)\left[\frac{\partial^2 g}{\partial x^2}(0,0,,\epsilon)\right]$$

from condition 2 we see that:

$$\frac{\partial^2 G}{\partial x^2}(0,0,\epsilon) \neq 0$$

Also form the proof of theorem (5.2) we can see:

$$\frac{\partial^2 G}{\partial x\partial \epsilon}(0,0,\epsilon) = (I-E)\left[\epsilon\left(\frac{\partial^2 g}{\partial x\partial \epsilon}(0,0,\epsilon)\right)\right]$$

and from condition 3 we see that:

$$\frac{\partial^2 G}{\partial x\partial \epsilon}(0,0,\epsilon) \neq 0$$

So the bifurcation equation (5.14) satisfy transcritical bifurcation conditions, and it is locally equivalent to one of the following normal forms:

$$\frac{d\eta}{dt} = \pm\mu\eta \pm \eta^2$$

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