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website: http:// jsci.utq.edu.iq Email: utjsci@utq.edu.iq

On $S\beta$ -continuous and $S^*\beta$ -continuous Functions

B. K. Mahmood

Salah Al –Deen`s Education - The Ministry of Education - Iraq

Abstract

In this paper we intoduce and investigate a new class of sets and functions called supra β- open sets , $S\beta$ - continuous functions and $S^*\beta$ - continuous .

حول الدوال الفوقية المستمرة والدوال المحيرة الفوقية المستمرة

بلقيس خليل محمود

وزارة التربية – تربية صلاح الدين

المستخلص

الهدف من هذا البحث هو تقديم صف جديد من المجموعات والدوال بين الفضاءات التبولوجية تسمى مجموعات بيتا المفتوحة الفوقية و الدوال الفوقية المستمرة والدوال المحيرة الفوقية المستمرة.

1- Introduction

In 1983 , A.S. Mashhour etal. [2] introduced the supra topological spaces and studied S - continuous functions and S^* - continuous functions . In 1987 , M. E. Abd El-Monsef etal. [1] introduced the fuzzy supra topological spaces and studies fuzzy supra-continuous functions and obtained some properties and characterizations. In 1996 , Keun [4] introduced fuzzy S - continuous , fuzzy S - open and fuzzy S - closed maps and established a number of characterizations. In 2008 , R. Devi etal. [3] introduced supra α - open sets and $S\alpha$ - continuous functions . Now we introduce the concept of supra β- open sets , $S\beta$ - continuous functions and $S^*\beta$ - continuous and investigate some of the basic properties for this class of functions .

2- Preliminaries and basic definitions:

All topological spaces considered in this paper lack any separation axioms unless explicitly stated. The topology of a space is denoted by $\Im$ and $(X,\Im)$ will be replaced by X if there is no chance for confusion. For any subset $A \subseteq X$, the closure and the interior of A in X with respect to $\Im$ are denoted by *cl(A)* and *int(A)* respectively . The complement of *A* is denoted by $X - A$.

Definition 1: [2] A sub family $\Im^*$ of X is said to be a supra topology on X if :
(1) $X , \ \phi \in \Im^*$

(2) If $A_i \in \Im^*$ for all $i \in J$, then $\bigcup A_i \in \Im^*$, $(X,\Im^*)$ is called a supra topological space . The elements of $\Im^*$ are called supra open sets in $(X,\Im^*)$ and complement of a supra open set is called a supra closed set.

Definition 2: [2] the supra clousre of a set A is denoted by supra *cl*(A) and defined as supra *cl(A)* = $\bigcap\{B : B$ is a supra closed and $A \subseteq B\}$.
The supra interior of a set A is denoted by supra *int*(A) and defined as supra *int*(A) = $\bigcup\{B : B$ is a $\Im$ supra open and $A \supseteq B\}$.

Definition 3: [2] Let $(X,\Im)$ be a topological space and $\Im^*$ be a supra topology on X. We call $\Im^*$ a supra topology associated with $\Im$ if $\Im \subset \Im^*$.

Definition 4: [2] Let $(X,\Im)$ and (Y,σ) be two topological spaces. Let $\Im^*$ and σ^* be associated supra topologies with $\Im$ and σ respectively. Let $f : X \to Y$ be a function from X into Y , then f is called a S - continuous function if the inverse image of each open set in Y is supra open in $(X,\Im^*)$.

Definition 5: [3] Let $(X,\Im^*)$ be a supra topological space. A set A is called supra semiopen set if $A \subseteq$ supra *cl(*supra *int(A))* .

Definition 6: Let $(X,\Im^*)$ be a supra topological space. A set A is called supra pre-open set if $A \subseteq$ supra *int(*supra *cl(A))* .

3- Basic properties of supra β - open sets:

In this section we introduce one new class of sets.

Definition 7: Let $(X,\Im^*)$ be a supra topological space. A subset A is called supra β - open set if $A \subseteq$ supra *cl(*supra *int(*supra *cl(A)))* .
And we notice that the complement of a supra β -open set is a supra β -closed set.

Theorem 3.1: Every supra open set is supra β - open set .
Proof:
Let A be a supra open set in $(X,\Im^*)$. Since $A \subseteq$ supra *cl(A)* , then supra *cl(A)* $\subseteq$ supra *cl(*supra *int(*supra *cl(A)))* . Hence $A \subseteq$ supra *cl(*supra *int(*supra *cl(A)))* .
$\therefore$ A is supra β - open set .

The converse of the above theorem is not true . This is shown by the following example .
Ex 1:
Let $(X,\Im^*)$ be a supra topological space , and X={ 1 , 2 , 3 } , $\Im^*$={ ϕ **,** X , {1} }**.** The set {1 , 2} is a supra β - open set , but not a supra open .

Theorem 3.2 : Every supra *pre*- open set is supra β - open set .
Proof :
Let A be a supra *pre*-open set in $(X,\Im^*)$, then $A \subseteq$ supra *int(*supra *cl(A))* .

Since supra *int(*supra *cl(A))* $\subseteq$ supra *cl(*supra *int(*supra *cl(A)))* , this implies that , $A \subseteq$ supra *cl(*supra *int(*supra *cl(A)))*

$\therefore$ A is supra β**-** open set .

Th converse of the last theorem is not true, we show that by the following example **.**

Ex 2 :

Let $(X, \Im^*)$ be a supra topological space , Where X={ 1 , 2 , 3 , 4 } and $\Im^*$ = { ϕ , X , {1} , {3 , 4 } , {1 , 3 , 4 } }. Here A = {2 , 3 , 4 } is a supra β**-** open set , but not a supra pre-open set .

In [3] the second part of the following theorem is not nesscery holding but in this paper is holding .

Theorem 3.3:

(i) Finite union of supra β-open sets is always a supra β-open set .

(ii) Finite intersection of supra β-open sets is always a supra β-open set

Proof :

(i) Let A and B be two supra β-open sets . Then

$A \subseteq$ supra *cl(*supra *int(*supra *cl(A)))* and $B \subseteq$ supra *cl(*supra *int(*supra *cl(B)))*

This implies , $A \cup B \subseteq$ supra *cl(*supra *int(*supra *cl(*$A \cup B$)))

$\therefore$ $A \bigcup B$ is a supra β-open set .

(ii) Let A and B be two supra β-open sets . Then

$A \subseteq$ supra *cl(*supra *int(*supra *cl(A)))* and $B \subseteq$ supra *cl(*supra *int(*supra *cl(B)))*

This implies, $A \cap B \subseteq$ supra *cl(*supra *int(*supra *cl(*$A \cap B$)))

$\therefore$ $A \cap B$ is a supra β-open set .

Corollary :

(i) Finite intersection of supra β-closed sets is always a supra β-closed set .

(ii) Finite union of supra β-closed sets is always a supra β-closed set .

Definition 8 : The supra β*-clousre* of a set A is denoted by supra β*-cl(A)* and defined as supra β*-cl(A)*= $\bigcap${B *:* B is a supra β-closed set and $A \subseteq B$ } .

The supra β*-interior* of a set A is denoted by supra β*-int(A)* and defined as supra β*-int(A)* =$\bigcup${B *:* B is a supra β-open set and $A \supseteq B$ } .

Remark : It is clear that supra β*-int(A)* is a supra β-open set and supra β*-cl(*A) is a supra β-closed set .

Theorem 3.4 :

(i) $X -$supra β*-int(*A) = supra β*-cl(* $X - A$*)* .

(ii) $X -$supra β*-cl(*A) = supra β*-int(* $X - A$*)* .

Proof : obvious

Theorem 3.5 : The following statements are true for every A and B sets :

(1) supra β*-int(A)* $\bigcup$ supra β*-int(B)* = supra β*-int(*$A \bigcup B$*).*

(2) supra β*-cl(A)* $\bigcap$ supra β*-cl(B)* = supra β*-cl(*$A \bigcap B$*).*

Proof :

(1) Let supra β-*int(A)* = $\bigcup\{C : C$ is a supra β-open set and $A \supseteq C\}$ and supra β-*int(B)* = $\bigcup\{D : D$ is a supra β-open set and $B \supseteq D\}$, then the union of these sets $=\bigcup\{C \bigcup D : C \bigcup D$ is a supra β-open set and $A \bigcup B \supseteq C \bigcup D\}$, if $K=C \bigcup D$, then supra β-*int(A)* $\bigcup$ supra β-*int(B)* = $\bigcup\{K : K$ is a supra β-open set and $A \bigcup B \supseteq K\}$.

(2) Let supra β-*cl(A)* = $\bigcap\{C : C$ is a supra β-closed set and $A \subseteq C\}$ and supra β-*cl(B)*= $\bigcap\{D : D$ is a supra β-closed set and $B \subseteq D\}$, then the intersection of these sets = $\bigcap\{C \bigcap D : C \bigcap D$ is a supra β-closed set and $A \bigcap B \subseteq C \bigcap D\}$, if $L=C \bigcap D$, then supra β-*cl(A)*$\bigcup$ supra β-*cl(B)* = $\{L : L$ is a supra β-closed set and $A \bigcap B \subseteq L\}$.

4 - $S\beta$ - continuous functions:

In this section we introduce a new class of function.

Definition 9 : Let $(X,\Im)$ and (Y,σ) be two topological spaces and $\Im^*$ be associated supra topology with $\Im$. We defined a function $f:(X,\Im)\to(Y,\sigma)$ to be a $S\beta$ - continuous function if the inverse image of each open set in Y is a $\Im^*$ - supra β - open set of X .

Theorem 4.1 : Every continuous function is $S\beta$ - continuous function .

Proof :

Let $f:(X,\Im)\to(Y,\sigma)$ be a continuous function . Therefore $f^{-1}(A)$ is open set in X for each open set A in Y . But $\Im^*$ is associated with $\Im$. That is mean $\Im \subset \Im^*$. This implies $f^{-1}(A)$ is a supra open in X . Since every supra open set is supra β - open set by (Th.3.1) . This implies $f^{-1}(A)$ is a $\Im^*$ - supra β - open in X .
Hence f is a $S\beta$ - continuous function .

Theorem 4.2 : Every S - continuous function is $S\beta$ - continuous function.

proof:

Let $f:(X,\Im)\to(Y,\sigma)$ be S - continuous function . Therefore $f^{-1}(A)$ is supra open set in X for each open set A in Y . But $\Im^*$ is associated with $\Im$. Since every supra open set is supra β - open set by (Th.3.1) . This implies $f^{-1}(A)$ is a $\Im^*$ - supra β - open in X .
Hence f is a $S\beta$ - continuous function .

The converse of the two theorems (4.1) and (4.2) may not be true . We can show that by the following example .

Ex 3 :

Let $X = \{1 , 2 , 3\}$ and $\Im = \{\phi, X, \{1 , 2\}\}$ be a topology on X . The supra topology $\Im^*$ is defined as follows , $\Im^* = \{\phi, X, \{1\}, \{1 , 2\}\}$. Let $f:(X,\Im)\to(X,\Im)$ where $f(1) = 1$, $f(2) = 3$, $f(3) = 2$.
Since $f^{-1}(\{1 , 2\}) = \{1 , 3\}$ is a supra β - open set in X . Then f is a $S\beta$ - continuous function . But the inverse image of { 1 , 2} is not open and not supra open set in X . So f is not S - continuous and not continuous function.

Theorem 4.3 : Let $(X,\Im)$ and (Y,σ) be two topological spaces . Let f be a function from X into Y . Let $\Im^*$ be associated supra topologies with $\Im$. Then the following are equivalent :

(1) f is $S\beta$ - continuous function .

(2) The inverse image of closed set in Y is supra β - closed set in X .

(3) Supra β -$cl(f^{-1}(A)) \subseteq f^{-1}(cl(A))$ for every set A in Y .

(4) f(supra β -$cl(A)) \subseteq cl(f(A))$ for every set A in X .

(5) $f^{-1}(int(B)) \subseteq$ supra β -$cl(f^{-1}(B))$ for every set B in Y .

proof :

$(1) \Rightarrow (2)$

Let A be a closed set in Y , then $Y - A$ is open in Y . Thus , $f^{-1}(Y - A) = X - f^{-1}(A)$ is supra β - open in X .It follows that $f^{-1}(A)$ is a supra β - closed set of X .

$(2) \Rightarrow (3)$

Let A be any subset of X . Since $cl(A)$ is closed in Y , then it follows that $f^{-1}(cl(A))$ is supra β - closed in X . Therefore

$f^{-1}(cl(A))=$ supra β -$cl(f^{-1}(cl(A))) \supseteq$ supra β -cl $(f^{-1}(A))$.

$(3) \Rightarrow (4)$

Let A be any subset of X . By (3) we obtain ,

$f^{-1}(cl(A)) \supseteq$ supra β -$cl(f^{-1}(f(A))) \supseteq$ supra β -cl (A) and hence

f (supra β -cl $(A)) \supseteq cl(f(A))$.

$(4) \Rightarrow (5)$

Let f(supra β -$cl(A)) \subseteq cl(f(A))$ for every set A in X . Then supra β -$cl(A) \subseteq f^{-1}(cl(f(A)))$, $X -$ supra β -cl $(A) \supseteq X - f^{-1}(cl(f(A)))$ and supra β -int $(X - A) \supseteq f^{-1}(int(Y - f(A)))$. Then supra β -$int(f^{-1}(B)) \supseteq f^{-1}(int(B))$. Therefore $f^{-1}(int(B)) \subseteq$ supra β -$int(f^{-1}(B))$, for every B in Y .

$(5) \Rightarrow (1)$

Let A be an open set in Y . Therefore , $f^{-1}(int(A)) \subseteq$ supra β -$int(f^{-1}(A))$, hence f^{-1} (A)$\subseteq \beta$ -$int(f^{-1}(A))$. But by other hand , we know that , supra β -int f^{-1} $(A)) \subseteq f^{-1}(A))$. Then f^{-1} (A) is a supra β - *open* set

Theorem 4.4 : If a function $f:(X,\Im) \rightarrow (Y,\sigma)$ is a $S\beta$ - continuous and $g:(Y,\sigma) \rightarrow (Z,\lambda)$ is continuous , then $(g \circ f)$ is $S\beta$ - continuous .

Proof :

Let A be an open set in Z . To prove that the inverse image of every open set in Z is a supra β -open set in X . Since g is continuous then $B = g^{-1}(A)$ is open set in Y , and since f *is* $S\beta$ - continuous then $C = f^{-1}(B)$ is a supra β - open in X . That is mean $(g \circ f)^{-1}(A) = (f^{-1} \circ g^{-1})(A) = f^{-1}(g^{-1}(A)) = f^{-1}(B) = C$.

$\therefore$ $(g \circ f)$ is a $S\beta$ - continuous function .

5 - $S^*\beta$ - continuous functions :

In this section we introduce a new class of function .

Definition 10 : Let $(X,\Im)$ and (Y,σ) be two topological spaces and $\Im^*$, σ^* are two associated supra topologies with $\Im$, σ .We defined a function $f:(X,\Im) \rightarrow (Y,\sigma)$ to be a $S\beta$ - continuous function if the inverse image of each σ^* -supra β -open set in Y is a $\Im^*$ - supra β - open set of X .

Theorem 5.1 : Every $S^*\beta$ - continuous function is $S\beta$ - continuous function .

Proof :

Let A be an open set in Y . Since every open set is supra β - open set and f is $S^*\beta$ - continuous , then f^{-1} (A) is supra β - open set in X .

$\therefore f$ is $S\beta$ - continuous function .

Theorem 5.2 : If a map $f:(X,\Im)\rightarrow(Y,\sigma)$ is a $S^*\beta$ - continuous and $g:(Y,\sigma)\rightarrow(Z,\lambda)$ is continuous , then $(g\circ f)$ is $S^*\beta$ - continuous .

Proof :

Let A be an open set in Z . We must prove that the inverse image of every supra β - open open set in Z is a supra β -open set in X .

Since g is continuous then $B=g^{-1}(A)$ is open set in Y , and since every open set is supra β - open set , f *is* $S^*\beta$ - continuous then $C=f^{-1}(B)$ is a supra β - open in X . That is mean

$(g\circ f)^{-1}(A)=(f^{-1}\circ g^{-1})(A)=f^{-1}(g^{-1}(A))=f^{-1}(B)=C$.

$\therefore$ $(g\circ f)$ is a $S^*\beta$ - continuous function .

Theorem 5.3: If a function $f:(X,\Im)\rightarrow(Y,\sigma)$ is a $S^*\beta$ - continuous and $g:(Y,\sigma)\rightarrow(Z,\lambda)$ is $S^*\beta$ - continuous , then $(g\circ f)$ is $S^*\beta$ - continuous .

Proof :

Let A be a supra β - open set set in Z . We must prove that the inverse image of every supra β - open open set in Z is a supra β -open set in X .

Since g is $S^*\beta$ - continuous then $B=g^{-1}(A)$ is supra β - open set in Y , and since f *is* $S^*\beta$ - continuous then $C=f^{-1}(B)$ is a supra β - open in X . That is mean

$(g\circ f)^{-1}(A)=(f^{-1}\circ g^{-1})(A)=f^{-1}(g^{-1}(A))=f^{-1}(B)=C$.

$\therefore$ $(g\circ f)$ is a $S^*\beta$ - continuous function .

References

(1) Abd El – Monsef , M.E and Ramadan ,A. E. , *On fuzzy supra topological spaces* , Indian J. Pure and Appl. Math. no.4 , 18(1987): 322-329 .

(2) Devi , R. , Sampathkumar , S. and Caldas , M. , On supra α - open sets and $S\alpha$ - continuous function , no.2 , 16(2008) : 77-84 .

(3) Mashhour , A. S. , Allam , A. A. ,Mahmoud , F. S. and Khedr , F. H. , *On supra topological spaces* , Indian J. Pure and Appl. Math. no.4 , 14(1983) : 502-510 .

(4) Won Keun Min , *On fuzzy s-continuous functions* , Kangweon- Kyungki Math. J. , no. 1 , 4(1996) : 77-82 .