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Generalized GN'–Function for n–Variable

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الملخص

في هذا العمل وسعنا نظرية الاعتيادية لفضاءات اورلسزالمتولدة بواسطة المتغيرات الحقيقية وبشكل خاص الدالة المتغيرات n عممناها لـ GN*–

ABSTRACT

In this work, we extend the usual theory of Orlicz spaces generated by real valued N-functions of a real variable. In particular, GN*-functions are the generalization of the variable N-functions used by Portnov and the non-decreasing -function by Wang.

Firstly, we begin with new definitions:

1.Introduction and Basic Concept

In what follows T will denote a space of point with σ -finite measure and E^n a dimensional Euclidean space.

Definition1.1[4]

Orlicz space $\mathcal{L}_{\mathcal{M}} = \mathcal{L}_{\mathcal{M}}(\Omega, \mu)$ is a Banach space consisting of all $f \in S(\Omega, \mu)$ *where* $S(\Omega, \mu)$ is a ring of all measurable functions on the space with bounded measure (Ω, μ).

such that $\int_{\Omega} M(|f|)d\mu < \infty,$

With the Luxemburg Nakano norm $\|f\|_M = \inf\{\lambda > 0 : \int_{\Omega} M(\frac{|f|}{\lambda})d\mu \leq 1\}$

Orlicz spaces Lm are natural generalization of Lp space, where $Lp(I)$ consists of all the measurable functions f defined on the interval I for which

$$(\int_I |f|^p)^{\frac{1}{p}} < \infty \qquad [1]$$

They have very rich topological and geometrical structures; they may possess peculiar properties that do not occur in an ordinary L_p space.

Definition1.2:[3]

Let $M : I \to R$ be defined on some interval of the real line R. A function *M* is called convex if

$$M(\frac{u_1+u_2}{2}) \leq \frac{1}{2}(M(u_1)+M(u_2)) \qquad (1.2.1)$$

for all $u_1, u_2 \in I$

Definition 1.3:[2]

Let $M(t,x,y)$ be a real valued non-negative function defined on $T \times E^n \times E^n$ such that:

(i) $M(t,x,y) = 0$ if and only if *x,y* are the zero vectors $x, y \in E^n$, $\forall t \in T$

(ii) $M(t,x,y)$ is a continuous convex function of x, y for each *t* and a measurable function of *t* for each x, y,

(iii) For each $t \in T$, $\lim\limits_{\substack{\|x\|=\infty \\ \|y\|=\infty}} \frac{M(t,x,y)}{\|x\|\|y\|} = \infty$, and

(iv)There are constants $d \geq 0$ and $d_1 \geq 0$ such that

$$\inf_t \inf_{\substack{c\geq d \\ c'\geq d_1}} k(t,c,c') > 0 \qquad (1.3.1)$$

Where

$$k(t,c,c') = \frac{\underline{M}(t,c,c')}{\overline{M}(t,c,c')},$$

$\overline{M}(t,c,c') = \sup\limits_{\substack{|x|=c \\ |y|=c'}} M(t,x,y), \underline{M}(t,c,c') = \inf\limits_{\substack{|x|=c \\ |y|=c'}} M(t,x,y)$ and if $d \rangle 0$ and $d_1 > 0$,

then $\overline{M}(t,d,d_1)$ is an integrable function of *t*. We call the function satisfying the properties (i)-(iv) a generalized N*-function or a GN*-function.

Definition 1.4:

Let $M(t,x_1,x_2,\ldots,x_n)$ be a real valued non-negative function defined on $T\times \underbrace{E^n\times E^n\times \ldots \times E^n}_{n-times}$ such that: (i) $M(t,x_1,x_2,\ldots,x_n)=0$ if and only if $x_1,x_2,\ldots,x_n$ are the zero vectors $x_1,x_2,\ldots,x_n\in E^n$, $\forall t\in T$

(ii) $M(t,x_1,x_2,\ldots,x_n)$ is a continuous convex function of $x_1,x_2,\ldots,x_n$ for each t and a measurable function of t for each $x_1,x_2,\ldots,x_n$,

(iii) For each $t\in T$, $\lim\limits_{\substack{\|x_1\|=\infty\\ \|x_2\|=\infty\\ \|x_n\|=\infty}} \dfrac{M(t,x_1,x_2,\ldots,x_n)}{\|x_1\|\|x_2\|\ldots\|x_n\|}=\infty$, and

(iv)There are constants $d_1\geq 0$, $d_2\geq 0,\ldots,d_n\geq 0$ such that

$$\inf_t \inf_{\substack{c_1\geq d_1\\ c_2\geq d_2\\ c_n\geq d_n}} k(t,c_1,c_2,\ldots,c_n)>0 \qquad (1.4.1)$$

Where

$$k(t,c_1,c_2,\ldots,c_n)=\frac{\underline{M}(t,c_1,c_2,\ldots,c_n)}{\overline{M}(t,c_1,c_2,\ldots,c_n)},$$

$$\overline{M}(t,x_1,c_2,\ldots,c_n)=\sup_{\substack{|x_1|=c_1\\ |x_2|=c_2\\ |x_n|=c_n}} M(t,x_1,x_2,\ldots,x_n), \qquad \underline{M}(t,c_1,c_2,\ldots,c_n)=\inf_{\substack{|x_1|=c_1\\ |x_2|=c_2\\ |x_n|=c_n}} M(t,x_1,x_2,\ldots,x_n)$$

and if $d_1>0$, $d_2>0,\ldots,$ $d_n>0$, then $\overline{M}(t,d_1,d_2,\ldots,d_n)$ is an integrable function of t. We call the function satisfying the properties (i)-(iv) a generalized N'-function or a GN'-function.

Example:

GN'-functions are coordinate independent and are not necessarily symmetric. Therefore, such functions as

$M(t,x_1,x_2,\ldots,x_n)$

$$M(t,x_1,x_2,\ldots,x_n)=(x_1+y_1)^2+(x_2+y_2)^2$$
$$+[(x_1+y_1)-(x_2+y_2)-\ldots-(x_n+y_n)]^2$$

which are nt non-decreasing (as defined in [3]) are allowed in the class of GN'-functions. The next theorem illustrates this point.

Theorem 1.5:[2]

If *M(t,x,y)*is a GN*-function and *A* is an orthogonal linear transformation defined on $E^n \times E^n$, with the range in $E^n \times E^n$, then $\tilde{M}(t,x,y) = M(t,Ax,Ay)$ is a GN*-function.

Theorem 1.6:**[2]**

A necessary and sufficient condition that (1.3.1) holds is that if $|x| \le |z|$ and $|y| \le |w|$, then there exists constants $K \ge 1, d \ge 0$ and $d' \ge 0$ such that $M(t,x,y) \le KM(t,z,w)$ for each *t* in *T*, $|x| \ge d$ and $|y| \ge d'$.

Remark:[2]

It is interesting to note that if $M(t,x,y)$ is a GN*-function, then $2\hat{M}(t,x,y) = M(t,x,y) + \tilde{M}(t,x,y)$ is also a GN*-function where $\tilde{M}(t,x,y)$ is defined as in Theorem 1.5. This means we can construct a symmetric (in *x* and *y*) GN*-function from one which does not possess this property.

For, if $\tilde{M}(t,x,y) = M(t,-x,-y)$, then $\hat{M}(t,x,y)$ is clearly symmetric in *x* and *y*.

Property (iv) of the definition 1.3 provides the condition which allows a natural gen eralization from GN*-function of a real variable to those of several real variables. Let us observe that the function $\overline{M}(t,c,c')$ is also a GN*-function of a real nonnegative variable *c* and *c'*. On the other hand, $M(t,c,c')$ need not even be convex in *c* and c'.

Since $\underline{M}(t,c,c') \le M(t,x,y) \le \overline{M}(t,c,c')$ for each *x* and *y* such that $|x| = c$ and $|y| = c'$ we would like to find a GN*-function which bounds

$\underline{M}(t,c,c')$ from below for all *c* and *c'* . If *d=0* and *d'=0* in Theorem (1.6), then $K^{-1}\overline{M}(t,c,c')$ would do.

One might accomplish the construction of such a function by taking the supremum of a class of convex functions bounding $\underline{M}(t,c,c')$ from below. This function would be convex. However, this class may be empty. The next theorem shows that this is not the case whenever $M(t,x,y)$ is a GN*-function. The construction employed can be applied to more general settings than those which exist here.

Theorem 1.7:[2]

If $M(t,x,y)$ is a GN*-function and $\underline{M}(t,c,c')$ is defined as above, then there exists a GN*-function $R(t,c,c')$ such that $R(t,c,c') \le \underline{M}(t,c,c')$ for all $c \ge 0$ and $c' \ge 0$.

2. Generalized GN'–Function

Theorem 2.1:

If $M(t,x_1,x_2,\ldots,x_n)$ is a GN'-function and A is an orthogonal linear transformation defined on $E^n \times E^n \times \ldots \times E^n$, with the range in $E^n \times E^n \times \ldots \times E^n$, then $\widetilde{M}(t,x_1,x_2,\ldots,x_n) = M(t,Ax_1,Ax_2,\ldots,Ax_n)$ is a GN'-function.

Proof:

Properties (i)-(iv) when applied to $\widetilde{M}(t,x_1,x_2,\ldots,x_n)$ follow immediately from the same properties for $M(t,x_1,x_2,\ldots,x_n)$ (see [5 ,Th 8.1]).

The next theorem characterizes a part of the property (iv) in the definition 1.4 and provides a means of comparing function values at different points for GN'-function when $|x_1|,|x_2|,\ldots,|x_n|$ are large.

Theorem 2.2:

A necessary and sufficient condition that (1.4.1) holds is that if $|x_1| \le |y_1|, |x_2| \le |y_2|, \ldots, |x_n| \le |y_n|$ then there exists constants $K \ge 1, d_1 \ge 0, d_2 \ge 0, \ldots, d_n \ge 0$ such that $M(t,x_1,x_2,\ldots,x_n) \le KM(t,y_1,y_2,\ldots,y_n)$ for each *t* in *T*, $|x_1| \ge d_1, |x_1| \ge d_1, \ldots, |x_n| \ge d_n$

Proof:

If (1.4.1) is true, then there exists constants $d_1 \ge 0, d_2 \ge 0, \ldots, d_n \ge 0$ such that $\tau(t) = \inf_{\substack{c_1 = d_1 \\ c_2 = d_2 \\ c_n = d_n}} k(t,c_1,c_2,\ldots,c_n) > 0$ for each *t* in *T*. By the definition of $K(t,c_1,c_2,\ldots,c_n)$ this means

$$M(t,x_1,x_2,..,x_n) \ge \underline{M}(t,|x_1|,|x_2|,\dots,|x_n|) \ge \tau(t)\overline{M}(t,|x_1|,|x_2|,\dots,|x_n|) \quad (2.2.1)$$

for any $x_1,x_2,..,x_n$ such that $|x_1| = c_1 \ge d_1, |x_2| = c_2 \ge d_2, \dots, |x_n| = c_n \ge d_n$.

On the other hand, if $d_1 \le |x_1| \le |y_1|, d_2 \le |x_2| \le |y_2|, \dots, d_n \le |x_n| \le |y_n|$ then the convexity of $M(t,x_1,x_2,..,x_n)$ and $M(t,0,0,\dots,0) = 0$ yields

$$\overline{M}(t,|y_1|,|y_2|,\dots,|y_n|) \ge \sup_{\substack{|z_1|=|x_1| \\ |z_2|=|x_2| \\ |z_n|=|x_n|}} M(t,z_1,z_2,\dots,z_n) \quad (2.2.2)$$

By combining (2.2.1) and (2.2.2), we arrive at

$$M(t,y_1,y_2,\dots,y_n) \ge \tau(t) \sup_{\substack{|z_1|=|x_1| \\ |z_2|=|x_2| \\ |z_n|=|x_n|}} M(t,z_1,z_2,\dots,z_n) \ge K^{-1} M(t,x_1,x_2,\dots,x_n),$$

When ever $d_1 \le |x_1| \le |y_1|, d_2 \le |x_2| \le |y_2|, \dots, d_n \le |x_n| \le |y_n|$ where $K^{-1} = \inf_t \tau(t) \rangle 0$.

The converse follows easily from the condition in the theorem.

Remark:

It is interesting to note that if $M(t,x_1,x_2,\dots,x_n)$ is a GN'-function, then $2\hat{M}(t,x_1,x_2,\dots,x_n) = M(t,x_1,x_2,\dots,x_n) + \tilde{M}(t,x_1,x_2,\dots,x_n)$ is also a GN'-function where $\tilde{M}(t,x_1,x_2,\dots,x_n)$ is defined as in Theorem 2.1. This means we can construct a symmetric (in $x_1,x_2,..,x_n$) GN'-function from one which does not possess this property.

For, if $\tilde{M}(t,x_1,x_2,\dots,x_n) = M(t,-x_1,-x_2,\dots,-x_n)$, then $\hat{M}(t,x_1,x_2,\dots,x_n)$ is clearly symmetric in $x_1,x_2,..,x_n$.

Property (iv) of the definition 1.4 provides the condition which allows a naturral generalization from N'-function of a real variable to those of several real variables. Let us observe that the function $\overline{M}(t,c_1,c_2,\dots,c_n)$ is also a GN'-function of a real nonnegative variable $c_1,c_2,\dots,c_n$.On the other hand, $M(t,c_1,c_2,\dots,c_n)$ need not even be convex in $c_1,c_2,\dots,c_n$.

Since $\underline{M}(t,c_1,c_2,\ldots,c_n) \le M(t,x_1,x_2,\ldots,x_n) \le \overline{M}(t,c_1,c_2,\ldots,c_n)$ for each $c_1,c_2,\ldots,c_n$ such that $|x_1| = c_1, |x_2| = c_2, \ldots, |x_n| = c_n$ we would like to find a GN'-function which bounds

$\underline{M}(t,c_1,c_2,\ldots,c_n)$ from below for all $c_1,c_2,\ldots,c_n$. If $d_1 = 0, d_2 = 0, \ldots, d_n = 0$ in Theorem (2.2), then $K^{-1}\overline{M}(t,c_1,c_2,\ldots,c_n)$ would do.

One might accomplish the construction of such a function by taking the supremum of a class of convex functions bounding $\underline{M}(t,c_1,c_2,\ldots,c_n)$ from below. This function would be convex. However, this class may be empty. The next theorem shows that this is not the case whenever $M(t,x_1,x_2,\ldots,x_n)$ is a GN'-function. The construction employed can be applied to more general settings than those which exist here.

Theorem 2.3:

If $M(t,x_1,x_2,\ldots,x_n)$ is a GN'-function and $\underline{M}(t,c_1,c_2,\ldots,c_n)$ is defined as above, then there exists a GN'-function $R(t,c_1,c_2,\ldots,c_n)$ such that $R(t,c_1,c_2,\ldots,c_n) \le \underline{M}(t,c_1,c_2,\ldots,c_n)$ for all $c_1 \ge 0, c_2 \ge 0, \ldots, c_n \ge 0$

Proof:

Since $\underline{M}(t,c_1,c_2,\ldots,c_n)$ satisfies property (iii) of the definition 1.4, given any $d > 0$ there are $c_1' > 0, c_2' > 0, \ldots, c_n' > 0$ such that $\underline{M}(t,c_1,c_2,\ldots,c_n) \ge d c_1 c_2 \ldots c_n$ whenever $c_1 \ge c_1', c_2 \ge c_2', \ldots, c_n \ge c_n'$. Let us define the function

$$P(t,c_1,c_2,\ldots,c_n) = \begin{cases} \sup\limits_{\substack{0<w_1\le 1\\ 0<w_2\le 1\\ 0<w_n\le 1\\ c_1w_1\ge c_1'\\ c_2w_2\ge c_2'\\ c_nw_n\ge c_n'}} \dfrac{\underline{M}(t,c_1w_1,c_2w_2,\ldots,c_nw_n)}{w_1w_2\ldots w_n} & if \quad c_1 \ge c_1', c_2 \ge c_2', \ldots, c_n \ge c_n' \\ \underline{M}(t,c_1,c_2,\ldots,c_n) & if \quad 0 \le c_1 < c_1', 0 \le c_2 < c_2', 0 \le c_n < c_n' \end{cases}$$

Then, it is easy to show that (i) $P(t,ac_1,ac_2,\ldots,ac_n)\le a^n P(t,c_1,c_2,\ldots,c_n)$ for $0\le a\le 1$,

(ii) $\left\{\frac{p(t,c_1,c_2,\ldots,c_n)}{c_1c_2\ldots c_n}\right\}$ is a non-decreasing function of $c_1,c_2,\ldots$ and c_n, and (iii) $P(t,c_1,c_2,\ldots,c_n)$ is finite for each $c_1,c_2,\ldots$ and c_n. We now obtain the desired function $R(t,c_1,c_2,\ldots,c_n)$ by defining

$$R(t,c_1,c_2,\ldots,c_n)=\int_0^{c_1}\int_0^{c_2}\ldots\int_0^{c_n}Q(t,s_1,s_2,\ldots,s_n)ds_1ds_2\ldots ds_n$$

where

$$Q(t,c_1,c_2,\ldots,c_n)=\begin{cases}\dfrac{P(t,c_1,c_2,\ldots,c_n)}{c_1c_2\ldots c_n} & if \quad c_1\ge c'_1,c_2\ge c'_2,\ldots,c_n\ge c'_n\\ \dfrac{c_1c_2\ldots c_n P(t,c'_1,c'_2,\ldots,c'_n)}{c_1'^2c_2'^2\ldots c_n'^2} & if \quad 0\le c_1<c'_1,0\le c_2<c'_2,0\le c_n<c'_n\end{cases}$$

Immediately we have

$$R(t,c_1,c_2,\ldots,c_n)\le c_1c_2\ldots c_nQ(t,c_1,c_2,\ldots,c_n)=P(t,c_1,c_2,\ldots,c_n)\le \underline{M}(t,c_1,c_2,\ldots,c_n).$$

It is not difficult to show that $R(t,c_1,c_2,\ldots,c_n)$ is also a GN'-function.

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