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L_n −proper Functions

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Abstract

In this paper, the concepts of L_n −proper Functions are defined depending on the definition of L −spaces which was presented by (**Kelley, J.C. : 1963 in [3]**) and the concept of Proper Functions which was presented by (**Bourbaki, N. in [1]**). Also we used this concept to study some theorms which are related to the concept of proper Functions.

1. Introduction:

Kelley, J.C. at 1963 in **[3]** presented a new concept, namely, Bitopological spaces which theirs members are sets depend upon two topologies in theirs definition on the same nonempty set.

Bitopological spaces which their members are L −open sets are called L −topological spaces or L −spaces. The concept of L −spaces are used in many principle topological concepts, for example Compactness, Connectedness, Separation axioms, Convergence and others. The concept of L_n −spaces is defined depending on the concept of L − spaces.

In this paper, the concept of L_n −***proper functions*** is defined depending on the concept of L_n −***spaces*** and the concept of ***proper functions*** which was presented by **Bourbaki, N. at** 1989 in **[1]**. Also we used this concept to study some theorms which are related to the concept of ***proper functions***.

2.1 Definitions [1], [2], [4].: Let X & Y be topological spaces and $f: X \to Y$ be a function. Then:

(i) f is called a **continuous function** if $f^{-1}(A)$ is an open set in X for every open set A in Y.

(ii) f is called an **open function** if $f(A)$ is an open set in Y for every open set A in X.

(iii) f is called a **closed function** if $f(A)$ is a closed set in Y for every closed set A in X.

2.2 Definition [1]: Let X and Y be topological spaces. Then the function $f: X \to Y$ is said to be proper function if :

(i) f is a continuous function.

(ii) $f \times i_z : X \times Z \to Y \times Z$ is a closed function for every topological space Z.

2.3 Proposition [1]: Let $f : X \to Y$ be a proper function. Then if $T \subseteq Y$, then the function $f_T : f^{-1}(T) \to T$ which agrees with f on $f^{-1}(T)$ is proper.

2.4 Definitions:

1. Let $\tau_1, \tau_2, \ldots, \tau_n$ $(n \geq 2)$ be topologies on a nonempty set X and $A, B \subseteq X$. Then

i. A is called an L_n **−open set** in X if there is a τ_1 −open set U in X such that $U \subseteq A \subseteq \bigcup_{i=2}^{n} \overline{U}^{\,i}$, where $\overline{U}^{\,i}$ is the *i-th* closure set of U in X *w.r.t.* τ_i $(2 \leq i \leq n)$.

The collection of all L_n −open sets in X is denoted by $L_n - O(X)$, and $(X, \tau_1, \tau_2, \ldots, \tau_n)$ is called an L_n **−topological space** or an L_n **−space** [for easiness written X is called an L_n **−space**], and the open sets in $(X, \tau_1, \tau_2, \ldots, \tau_n)$ are L_n −open sets in X .

ii. B is called an L_n **−closed set** in X if and only if B^c is an L_n −open set in X .

The collection of all L_n −closed sets in X is denoted by $L_n - C(X)$.

2. **The L_n −product space** of L_n −spaces $(X, \tau_1, \tau_2, \ldots, \tau_n)$ & $(Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ is the L_n −space $(X \times Y, \mu_1, \mu_2, \ldots, \mu_n)$, where μ_1, μ_2, ….., μ_n are the product

topologies on $X \times Y$ induced by $(\tau_1 \,\&\, \tau'_1)$, $(\tau_2 \,\&\, \tau'_2)$, . .$(\tau_n \,\&\, \tau'_n)$ respectively.

3. **The L_n −subspace** A of an L_n −space $(X, \tau_1, \tau_2, \ldots, \tau_n)$ is the L_n −space $(A, \tau_{1A}, \tau_{2A}, \ldots, \tau_{nA})$, where $\tau_{1A}, \tau_{2A}, \ldots, \tau_{nA}$ are the relative topologies of $\tau_1, \tau_2, \ldots, \tau_n$ respectively on A in X .

4. Let $(X, \tau_1, \tau_2, \ldots, \tau_n)$ & $(Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ be L_n −spaces, and let $f : (X, \tau_1, \tau_2, \ldots, \tau_n) \to (Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ be a function. Then:

(i) f is called L_n **−continuous** if $f^{-1}(A)$ is an L_n −open set in X for all $A \in \tau'_1$.

(ii) f is called L_n *−**continuous** if $f^{-1}(A)$ is an L_n −open set in X for all $A \in L_n - O(Y)$.

(iii) f is called L_n **−open** if $f(A)$ is an L_n −open set in Y for all $A \in \tau_1$.

(iv) f is called L_n *−**open** if $f(A)$ is an L_n −open set in Y for all $A \in L_n - O(X)$.

(v) f is called L_n-**closed** if $f(A)$ is an L_n-closed set in Y for every τ_1-closed set A in X.

(vi) f is called L_n*-**closed** if $f(A)$ is an L_n-closed set in Y for every L_n-closed set A in X.

(vii) The bijective function f is called L_n-**homeomorphism** if f is L_n-open (L_n-closed) and L_n-continuous.

(viii) The bijective function f is called L_n*-**homeomorphism** if f is L_n*-open (L_n*-closed) and L_n*-continuous.

5. The L_n-space X is said to be L_n-**homeomorphic** to the L_n-space Y if there is an L_n-homeomorphism of X on to Y,[written $X \overset{L_n}{\approx} Y$].

6. The L_n-space X is said to be L_n*-**homeomorphic** to the L_n-space Y if there is an L_n*-homeomorphism of X on to Y, [written $X \overset{L_n*}{\approx} Y$]

<u>2.5 Remarks:</u>

1. Let $(X,\tau_1,\tau_2,\ldots\ldots,\tau_n)$ be an L_n-space. Then

i. τ_1 is a sub collection of $L_n-O(X)$, (since $U \subseteq U \subseteq \bigcup_{i=2}^{n} \overline{U}^{\,i}$ for all $U \in \tau_1$).

ii. If $n1 < n2 \le n$, then $L_{n1}-O(X) \subseteq L_{n2}-O(X)$, since if $A \in L_{n1}-O(X)$ $\Rightarrow \exists U \in \tau_1 \ni U \subseteq A \subseteq \bigcup_{i=2}^{n1} \overline{U}^{\,i}$ & $\bigcup_{i=2}^{n1} \overline{U}^{\,i} \subseteq \bigcup_{i=2}^{n2} \overline{U}^{\,i}$ $\Rightarrow U \subseteq A \subseteq \bigcup_{i=2}^{n2} \overline{U}^{\,i}$ $\Rightarrow A \in L_{n2}-O(X)$ $\Rightarrow L_{n1}-O(X) \subseteq L_{n2}-O(X)$. Moreover $L_2-O(X) \subseteq L_3-O(X) \subseteq \ldots\ldots L_n-O(X)$.

iii. If $\tau_1 = D$ is the discrete topology, then $L_n-O(X) = \tau_1 = D$.

iv. If $\tau_1 = \{\phi, X\}$ is the indiscrete topology, then $L_n-O(X) = \tau_1 = \{\phi, X\}$.

v. If $\tau_i = D$ is the discrete topology for all $(2 \le i \le n)$, then $L_n-O(X) = \tau_1$.

vi. If U is $\bigcap_{i=1}^{n}\tau_i-$open in X & A is L_n-open in X, then $U \cap A$ is an L_n-open set in X, (since if $A \in L_n-O(X)$. $\Rightarrow \exists V \in \tau_1 \ni V \subseteq A \subseteq \bigcup_{i=2}^{n} \overline{V}^{\,i}$ $\Rightarrow U \cap V \subseteq U \cap A \subseteq U \cap \bigcup_{i=2}^{n} \overline{V}^{\,i} \subseteq \bigcup_{i=2}^{n} \overline{U \cap V}^{\,i}$ by using (***v***). Since $U \in \bigcap_{i=1}^{n}\tau_i$ & $V \in \tau_1$. $\Rightarrow U \cap V \in \tau_1$. $\Rightarrow U \cap A \in L_n-O(X)$).

vii. If A is $\bigcap_{i=1}^{n}\tau_i-$open in X and $B \in L_n-O(X)$, then $A \cap B \in L_n-O(A)$. (since if

$B \in L_n - O(X) . \Rightarrow \exists V \in \tau_1 \ni V \subseteq B \subseteq \bigcup_{i=2}^{n} \overline{V}^{\,i} . \Rightarrow V \cap A \subseteq$

$B \cap A \subseteq A \cap \bigcup_{i=2}^{n} \overline{V}^{\,i} \subseteq \bigcup_{i=2}^{n} \overline{V \cap A}^{\,i}$ by using *(v)*.

Since $\overline{V \cap A}^{\,i} \cap A = \overline{V \cap A}^{\,iA} . \Rightarrow V \cap A \subseteq B \cap A \subseteq A \cap \bigcup_{i=2}^{n} \overline{V}^{\,i} \subseteq \bigcup_{i=2}^{n} \overline{V \cap A}^{\,iA}$.

Since $V \in \tau_1 \Rightarrow V \cap A \in \tau_{1A} \Rightarrow B \cap A \in L_n - O(A)$).

2. Let $(X, \tau_1, \tau_2, \ldots, \tau_n)$ & $(Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ be $L_n -$spaces, and let $f:(X, \tau_1, \tau_2, \ldots, \tau_n) \to (Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ be a function. Then if $f:(X, \tau_1) \to (Y, \tau'_1)$ is a continuous [open,closed] function then $f:(X, \tau_1, \tau_2, \ldots, \tau_n) \to (Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ is an $L_n -$continuous [$L_n -$open , $L_n -$closed] function respectively.

<u>3.1 Definition:</u> Let X and Y be $L_n -$spaces and $f : X \to Y$ be a function. Then f is called an $L_n -$**proper function** if:

(i) f is an $L_n -$continuous function.

(ii) $f \times i_Z : X \times Z \to Y \times Z$ is $L_n -$closed for every $L_n -$space Z.

<u>3.2 Proposition:</u> Let $(X, \tau_1, \tau_2, \ldots, \tau_n)$ & $(Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ be $L_n -$spaces such that the function $f:(X, \tau_1) \to (Y, \tau'_1)$ is proper. Then $f:(X, \tau_1, \ldots, \tau_n) \to (Y, \tau'_1, , \ldots, \tau'_n)$ is an $L_n -$proper function for every topologies $\tau_i \,\&\, \tau'_i$, $(2 \le i \le n)$ on $X \,\&\, Y$ respectively.

<u>Proof:</u>

$f:(X, \tau_1, \tau_2, \ldots, \tau_n) \to (Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ is an $L_n -$continuous by using (**2.5**).

To prove that $f \times i_Z$ is $L_n -$closed, let $(Z, \eta_1, \eta_2, ..\eta_n)$ be an $L_n -$space. $\Rightarrow (Z, \eta_1)$ is a topological space. Since $f:(X, \tau_1) \to (Y, \tau'_1)$ is a proper function $\Rightarrow f \times i_Z : (X \times Z, \mu_1) \to (Y \times Z, \mu'_1)$ is a closed function for every topological space $(Z, \eta_1) \Rightarrow f \times i_Z : (X \times Z, \mu_1, \mu_2, \ldots, \mu_n) \to (Y \times Z, \mu'_1, \mu'_2, \ldots, \mu'_n)$ is an $L_n -$closed function for every $L_n -$space $(Z, \eta_1, \eta_2, ..\eta_n)$ by using **(2.5)**, where $(X \times Z, \mu_1, \mu_2, .., \mu_n)$ and $(Y \times Z, \mu'_1, \mu'_2, \ldots, \mu'_n)$ are the $L_n -$product spaces of $[(X, \tau_1, \tau_2, \ldots, \tau_n)$ & $(Z, \eta_1, \eta_2, ...\eta_n)]$ and $[(Y, \tau'_1, \tau'_2, ..., \tau'_n)$ & $(Z, \eta_1, \eta_2, ...\eta_n)]$ respectively. $\Rightarrow f:(X, \tau_1, \tau_2, \ldots, \tau_n) \to (Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ is an $L_n -$proper function.

<u>3.3 Proposition:</u> Let $f:(X, \tau_1, \tau_2, \ldots, \tau_n) \to (Y, \tau'_1, \tau'_2, \ldots, \tau'_n)$ be an $L_n -$proper function. Then if T is a subset of Y such that $T \in \bigcap_{i=1}^{n} \tau'_i$ and

$f:(X,\tau_1)\to(Y,\tau'_1)$ is a continuous function, then the function $f_T:(f^{-1}(T),\tau_{1f^{-1}(T)},\dots,\tau_{nf^{-1}(T)})\to(T,\tau'_{1T},\dots,\tau'_{nT})$ which agrees with f on T is an L_n-proper function.

Proof:

To prove that f_T is an L_n-continuous function. Since $f:(X,\tau_1)\to(Y,\tau'_1)$ is a continuous function. $\Rightarrow f_T:(f^{-1}(T),\tau_{1f^{-1}(T)})\to(T,\tau'_{1T})$ is a continuous function by using **(2.3).** $\Rightarrow f_T:(f^{-1}(T),\tau_{1f^{-1}(T)},\tau_{2f^{-1}(T)},\dots,\tau_{nf^{-1}(T)})\to(T,\tau'_{1T},\tau'_{2T},\dots,\tau'_{nT})$ is an L_n-continuous function by using **(2.5)**.

To prove that $(f_T\times i_Z)$ is an L_n-closed function, let $(Z,\eta_1,\eta_2,..\eta_n)$ be an L_n-space. Let $(X\times Z,\mu_1,\mu_2,..,\mu_n)$ & $(Y\times Z,\mu'_1,\mu'_2,...,\mu_n)$ are the L_n-product spaces of $[(X,\tau_1,\tau_2,...,\tau_n)$ & $(Z,\eta_1,\eta_2,...\eta_n)]$ & $[(Y,\tau'_1,\tau'_2,...,\tau'_n)$ & $(Z,\eta_1,\eta_2,...\eta_n)]$ respectively.

Since $(f_T\times i_Z)=(f\times i_Z)_{T\times Z}$.

$\Rightarrow(f_T\times i_Z):((f\times i_Z)^{-1}(T\times Z),\mu_{1(f\times i_Z)^{-1}(T\times Z)},\mu_{2(f\times i_Z)^{-1}(T\times Z)},\dots,\mu_{n(f\times i_Z)^{-1}(T\times Z)})\to(T\times Z,\mu'_{1T\times Z},\mu'_{2T\times Z},\dots,\mu'_{nT\times Z})$

Let B be a $\mu_{1(f\times i_Z)^{-1}(T\times Z)}-$closed in $(f\times i_Z)^{-1}(T\times Z).\Rightarrow B=(f\times i_Z)^{-1}(T\times Z)\cap A$, $A^c\in\mu_1\Rightarrow(f_T\times i_Z)(B)=(T\times Z)\cap(f\times i_Z)(A)$. Since f is an L_n-proper function. $\Rightarrow(f\times i_Z)$ is an L_n-closed function. $\Rightarrow(f\times i_Z)(A)\in L_n-C(Y\times Z)$.

Let $(f\times i_Z)(A)=F\Rightarrow(f_T\times i_Z)(B)=(T\times Z)\cap F$. Since $(T\times Z)-[(T\times Z)\cap F]=[(Y\times Z)-F]\cap(T\times Z)$, $T\in\bigcap_{i=1}^{n}\tau'_i$ & $Z\in\bigcap_{i=1}^{n}\eta_i$. $\Rightarrow$ $T\times Z\in\bigcap_{i=1}^{n}\mu'_i$ by **(2.5)**. Since $F\in L_n-C(Y\times Z)$. $\Rightarrow[(Y\times Z)-F]\in L_n-O(Y\times Z)$ $\Rightarrow$ $[(Y\times Z)-F]\cap(T\times Z)\in L_n-O(T\times Z)$ by **(2.5)** $.\Rightarrow(T\times Z)\cap F\in L_n-C(T\times Z)$. $\Rightarrow(f_T\times i_Z)(B)\in L_n-C(T\times Z).\Rightarrow(f_T\times i_Z)$ is an L_n-closed function. $\Rightarrow f_T$ is an L_n-proper function.

3.4 Examples:

(i) Let (X,τ_1) be a topological space and F be τ_1-closed in X. Then the function $i:(F,\tau_{1F})\to(X,\tau_1)$ defined by $i(x)=x$ for all $x\in F$, is a proper function, therefore

$i:(F,\tau_{1F},\tau_{2F},\ldots,\tau_{nF})\to(X,\tau_1,\tau_2,\ldots,\tau_n)$ is an L_n-proper function for every topologies τ_i , $(2\le i\le n)$ on X by **(3.2).**

(ii) Let (X,T) be a topological space and $I_X:(X,T)\to(X,T)$ such that $I_X(x)=x$, for all $x\in X$. Then I_X is a proper function, therefore $I_X:(X,\tau_1,\tau_2,\ldots,\tau_n)\to(X,\tau_1,\tau_2,\ldots,\tau_n)$ is an L_n-proper function for every topologies τ_i , $(2\le i\le n)$ on X by **(3.2)**.

(iii) Let u be the usual topology on $\Re$, and let $[a,b)\subseteq\Re, a<b.$ Then the function $i:([a,b),u_{[a,b)},\ldots,u_{[a,b)})\to(\Re,u,\ldots,u)$ ***(n-times)*** defined by $i(x)=x$ for all $x\in[a,b)$, is an L_n-proper function, but $i:([a,b),u_{[a,b)})\to(\Re,u)$ is not a proper function, (since it is not a closed function **[1]**).

(iv) Let *n*=2 and let u be the usual topology on $\Re$, and let $f:\Re\to\Re$ be the function such that $f(x)=0$ for all $x\in\Re$. Then $f:(\Re,u)\to(\Re,u)$ is a closed and continuous function, therefore $f:(\Re,u,u)\to(\Re,u,u)$ is an L_n-closed and an L_n-continuous function. But $f:(\Re,u,u)\to(\Re,u,u)$ is not L_n-proper function, since $(f\times i_{\Re}):(\Re^2,U,U)\to(\Re^2,U,U)$ defined by $(f\times i_{\Re})(x,y)=(0,y)$ for all $(x,y)\in\Re^2$, is not an L_n-closed function, where U is the usual topology on $\Re^2$. Since $K=\{(x,y)\in\Re^2/xy=1\}$ is $U-$closed in $\Re^2$ and $(f\times i_{\Re})(K)=\{0\}\times\Re\setminus\{0\}\notin L_n-C(\Re\times\Re)$. Since $\{0\}\times\Re\setminus\{0\}\overset{L_n}{\approx}\Re\setminus\{0\}$ & $\overline{\Re\setminus\{0\}}^{L_n}=\Re$, therefore $f:(\Re,u,u)\to(\Re,u,u)$ is not an L_n-proper function. Therefore $f:(\Re,u)\to(\Re,u)$ is not proper function by **(3.2)**.

<u>3.5 Definitions*:*</u> Let $X\,\&\,Y$ be L_n-spaces and $f:X\to Y$ be a function. Then:

$(i)\,f$ is called an L_n*-**proper function** if:

1. f is an L_n-continuous function.

2. $(f\times i_Z)$ is an L_n*-closed function for every L_n-space Z.

$(ii)\,f$ is called an L_n**-**proper function** if:

1. f is an L_n*-continuous function.

2. $(f\times i_Z)$ is an L_n*-closed function for every L_n-space Z.

Notice that: f is an L_n**-**proper function** ⟶ f is an L_n*-**proper function**

↓ f is an L_n-**proper function**

(f is an L_n**-**proper function** ⟶ f is an L_n-**proper function**)

<u>3.6 Proposition*:*</u> Let $f:(X,\tau_1,\ldots,\tau_n)\to(Y,\tau'_1,\ldots,\tau'_n)$ & $g:(Y,\tau'_1,\ldots,\tau'_n)\to(W,\tau''_1,\ldots,\tau''_n)$ be L_n-continuous functions. Then:

(i) If $f \& g$ are L_n-proper & L_n*-proper function respectively, and if f is an L_n*-continuous function, then $g \circ f$ is an L_n-proper function.

(ii) If $f \& g$ are L_n-proper & L_n*-proper respectively, and if $g:(Y,\tau'_1) \to (W,\tau_1'')$ is a continuous function, then $g \circ f$ is an L_n-proper function.

(iii) If $g \circ f$ is an L_n-proper function, f is surjective and $f:(X,\tau_1) \to (Y,\tau'_1)$ is a continuous function , then g is an L_n-proper function.

(iv) If $g \circ f$ is an L_n-proper function, and if g is an injective L_n*-continuous function, then f is an L_n-proper function.

Proof:

(i)

To prove that $g \circ f$ is L_n-continuous. Let $V \in \tau''_1$. Since g is L_n-continuous $\Rightarrow g^{-1}(V) \in L_n - O(Y)$. Since f is L_n*-continuous $\Rightarrow$ $f^{-1}(g^{-1}(V))$ = $(g \circ f)^{-1}(V) \in L_n - O(X) \Rightarrow g \circ f$ is an L_n-continuous function by **(2.4)**.

To prove that $(g \circ f) \times i_Z$ is L_n-closed. Let $(Z,\eta_1,\eta_2,...\eta_n)$ be an L_n-space. Observe that $(g \circ f) \times i_Z = (g \times i_Z) \circ (f \times i_Z)$.

$$(X \times Z, \mu_1,...,\mu_n) \xrightarrow{f \times i_Z} (Y \times Z, \mu'_1,...,\mu'_n) \xrightarrow{g \times i_Z} (W \times Z, \mu''_1,...,\mu''_n)$$

where $(X \times Z, \mu_1,..., \mu_n), (Y \times Z, \mu'_1,..., \mu'_n)$ & $(W \times Z, \mu''_1,..., \mu''_n)$ are the L_n-product spaces of $[(X,\tau_1,\tau_2,...,\tau_n)$ & $(Z,\eta_1,\eta_2,..\eta_n)]$, $[(Y,\tau'_1,\tau'_2,...,\tau'_n)$ & $(Z,\eta_1,\eta_2,..\eta_n)]$ & $[(W,\tau''_1,\tau''_2,...,\tau''_n)$ & $(Z,\eta_1,\eta_2,...\eta_n)]$ respectively. Let F be μ''_1-closed. Since f & g are L_n-proper & L_n*-proper respectively. $\Rightarrow f \times i_Z$ & $g \times i_Z$ are L_n-closed & L_n*-closed respectively. $\Rightarrow ((g \times i_Z) \circ (f \times i_Z))(F)$ is an L_n-closed set in $W \times Z$ by **(2.4)** $\Rightarrow (g \times i_Z) \circ (f \times i_Z)$ is an L_n-closed function by **(2.4)** $\Rightarrow (g \circ f) \times i_Z$ is an L_n-closed function. $\Rightarrow g \circ f$ is an L_n-proper function.

(ii)

To prove that $g \circ f$ is L_n-continuous. Let $V \in \tau''_1 . \Rightarrow g^{-1}(V) \in \tau'_1$(since $g:(Y,\tau'_1) \to (W,\tau_1'')$ is continuous) $\Rightarrow g^{-1}(V) \in \tau'_1$. Since f is L_n-continuous $\Rightarrow f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \in L_n - O(X) \Rightarrow g \circ f$ is an L_n-continuous function by **(2.4)**.

To prove that $(g \circ f) \times i_Z$ is L_n-closed. Let $(Z,\eta_1,\eta_2,..\eta_n)$ be an L_n-space. Observe that $(g \circ f) \times i_Z = (g \times i_Z) \circ (f \times i_Z)$.

$$(X \times Z, \mu_1,..., \mu_n) \xrightarrow{f \times i_Z} (Y \times Z, \mu'_1,..., \mu'_n) \xrightarrow{g \times i_Z} (W \times Z, \mu''_1,..., \mu''_n)$$

where $(X\times Z,\mu_1,...,\mu_n),(Y\times Z,\mu'_1,...,\mu'_n)\&(W\times Z,\mu''_1,...,\mu''_n)$ are the L_n-product spaces of $[(X,\tau_1,\tau_2,...,\tau_n)$ & $(Z,\eta_1,\eta_2,..\eta_n)]$, $[(Y,\tau'_1,\tau'_2,...,\tau'_n)$ & $(Z,\eta_1,\eta_2,..\eta_n)]$ & $[(W,\tau''_1,\tau''_2,...,\tau''_n)$ & $(Z,\eta_1,\eta_2,..\eta_n)]$ respectively. Let F be μ''_1-closed. Since f & g are L_n-proper & L_n*-proper respectively. $\Rightarrow f\times i_Z$ & $g\times i_Z$ are L_n-closed & L_n*-closed respectively. $\Rightarrow((g\times i_Z)\circ(f\times i_Z))(F)$ is an L_n-closed set in $W\times Z$ by **(2.4)** $\Rightarrow(g\times i_Z)\circ(f\times i_Z)$ is an L_n-closed function by **(2.4)** $\Rightarrow(g\circ f)\times i_Z$ is an L_n-closed function. $\Rightarrow g\circ f$ is an L_n-proper function.

(iii)

g is an L_n-continuous function.

Let $(Z,\eta_1,\eta_2,..\eta_n)$ be an L_n-space, to prove that $(g\times i_Z)$ is L_n-closed. Observe that $(g\circ f)\times i_Z=(g\times i_Z)\circ(f\times i_Z)$

$$(X\times Z,\mu_1,\dots,\mu_n)\xrightarrow{f\times i_Z}(Y\times Z,\mu'_1,\dots,\mu'_n)\xrightarrow{g\times i_Z}(W\times Z,\mu''_1,\dots,\mu''_n)$$

where $(X\times Z,\mu_1,...,\mu_n),(Y\times Z,\mu'_1,...,\mu'_n)\&(W\times Z,\mu''_1,...,\mu''_n)$ are the L_n-product spaces of $[(X,\tau_1,\tau_2,...,\tau_n)$ & $(Z,\eta_1,\eta_2,..\eta_n)]$, $[(Y,\tau'_1,\tau'_2,...,\tau'_n)$ & $(Z,\eta_1,\eta_2,..\eta_n)]$ & $[(W,\tau''_1,\tau''_2,...,\tau''_n)$ & $(Z,\eta_1,\eta_2,...\eta_n)]$ respectively. Since f is a surjective function. $\Rightarrow(f\times i_Z)$ is a surjective function.

Let A be a μ'_1-closed set in $Y\times Z$. To prove that $(g\times i_Z)(A)\in L_n-C(W\times Z)$. Since $(f\times i_Z)$ is a surjective function and $f:(X,\tau_1)\to(Y,\tau'_1)$ is a continuous function $\Rightarrow(f\times i_Z)$ is a continuous function $\Rightarrow\exists B^c\in\mu_1\ni A=(f\times i_Z)(B)$ by **(2.1)** . Since $(g\circ f)$ is an L_n-proper function. $\Rightarrow[(g\circ f)\times i_Z](B)\in L_n-C(W\times Z)$. Since $[(g\circ f)\times i_Z](B)$ = $(g\times i_Z)((f\times i_Z)(B))$ $=(g\times i_Z)(A)$. $\Rightarrow(g\times i_Z)(A)\in L_n-C(W\times Z)$ $\Rightarrow g$ is an L_n-proper function.

(iv)

f is an L_n-continuous function.

Let $(Z,\eta_1,\eta_2,...\eta_n)$ be an L_n-space. To prove that $(f\times i_Z)$ is an L_n-closed function. Observe that $(g\circ f)\times i_Z=(g\times i_Z)\circ(f\times i_Z)$

$$(X\times Z,\mu_1,...,\mu_n)\xrightarrow{f\times i_Z}(Y\times Z,\mu'_1,...,\mu'_n)\xrightarrow{g\times i_Z}(W\times Z,\mu''_1,...,\mu''_n)$$

where $(X\times Z,\mu_1,...,\mu_n),(Y\times Z,\mu'_1,...,\mu'_n)\&(W\times Z,\mu''_1,...,\mu''_n)$ are the L_n-product spaces of $[(X,\tau_1,\tau_2,...,\tau_n)$ & $(Z,\eta_1,\eta_2,...\eta_n)]$, $[(Y,\tau'_1,\tau'_2,...,\tau'_n)$

& $(Z,\eta_1,\eta_2,..\eta_n)]$ & $[(W,\tau''_1,\tau''_2,...,\tau''_n)$ & $(Z,\eta_1,\eta_2,...\eta_n)]$ respectively. Since g is an injective function. $\Rightarrow (g\times i_Z)$ is an injective function. Let A be a μ_1-closed set in $X\times Z$. Since $(g\circ f)$ is an L_n-proper function $\Rightarrow (g\circ f)\times i_Z$ is an L_n-closed function $\Rightarrow [(g\times i_Z)\circ(f\times i_Z)](A)\in L_n-C(W\times Z)$. Since $(g\times i_Z)$ is an injective function $\Rightarrow (f\times i_Z)(A)=(g\times i_Z)^{-1}[(g\times i_Z)\circ(f\times i_Z)](A)$. Since g is an L_n*-continuous function. $\Rightarrow (g\times i_Z)$ is L_n*-continuous. $\Rightarrow$ $(f\times i_Z)(A)$ = $(g\times i_Z)^{-1}[(g\times i_Z)\circ(f\times i_Z)](A)\in L_n-C(Y\times Z)$.

$\Rightarrow (f\times i_Z)$ is an L_n-closed function. $\Rightarrow f$ is an L_n-proper function.

3.7 Proposition: Let $f_1: X_1\to X_2$ & $f_2: X_2\to X_2$ be L_n-proper and L_n*-proper functions respectively. Then $f_1\times f_2: X_1\times X_2\to Y_1\times Y_2$ is an L_n-proper function.

Proof:

To prove that $f_1\times f_2$ is L_n-continuous function. Since f_1 & f_2 are L_n-continuous functions . $\Rightarrow f_1\times f_2$ is L_n-continuous function.

To prove that $(f_1\times f_2\times i_Z)$ is an L_n-closed function, let Z be an L_n-space. Consider

$$X_1\times X_2\times Z \xrightarrow{f_1\times i_{X_2}\times i_Z} Y_1\times X_2\times Z \xrightarrow{i_{Y_1}\times f_2\times i_Z} Y_1\times Y_2\times Z$$

$$(f_1\times f_2\times i_Z)=(i_{Y_1}\times f_2\times i_Z)\circ(f_1\times i_{X_2}\times i_Z)$$

Since f_1 is an L_n-proper function. $\Rightarrow (f_1\times i_{X_2}\times i_Z)=f_1\times i_{(X_2\times Z)}$ is an L_n-closed function. Since f_2 is an L_n*-proper function. $\Rightarrow (i_{Y_1}\times f_2\times i_Z)$ is an L_n*-closed function. $\Rightarrow (f_1\times f_2\times i_Z)=(i_{Y_1}\times f_2\times i_Z)\circ(f_1\times i_{X_2}\times i_Z)$ is an L_n-closed function by **(2.4)**. $\Rightarrow f_1\times f_2$ is an L_n-proper function.

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الخلاصة

في هذا البحث عُرف مفهوم الدوال السديدة من النمط L_n- بالاعتماد على مفهوم $L-space$ الذي قدمه العالم (***Kelley, J.C.***) سنة ١٩٦٣ في المصدر [3]) وعلى مفهوم الدوال السديدة الذي قدمه العالم (***Bourbaki, N.***) في المصدر [1]) . كذلك أستخدم هذا المفهوم لدراسة بعض المبرهنات التي تتعلق بمفهوم الدوال السديدة.