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Homotopy perturbation method with modified Riemann –Liouville derivative for solving fractional- time partial differential equations

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Abstract:

The aim of this paper is to present an efficient and reliable treatment of the Homotopy perturbation method (HPM) and the modified Riemann –Liouville derivative for partial differential equations with fractional time derivative. The fractional derivative is described in the Jumarie sense. The results reveal that the proposed method is very effective and simple and leads to accurate, approximately convergent solutions to fractional partial differential equations.

1-Introduction:

Differential equations of fractional order have been found to be effective to describe some physical phenomena such as rheology, damping laws, fluid flow and so on [1-3]. A review of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [1]. Different fractional partial differential equations have been studied and solved including the space –time fractional diffusion – wave equation [4, 5].

Recently, a new modified Riemann –Liouville left derivative is proposed by G.jumarie [6] Comparing with the classical caputo derivative, the definition of the fractional derivative is not required to satisfy higher integer-order derivative than α . Secondly αth derivative of a constant is zero. For these merits, G.Jumarie's modified derivative was successfully applied in the probability calculus [7], fractional Laplace problems [8].

The Homotopy perturbation method was established by Ji-Huan He in1999[10].The method has been used by many authors to handle a wide variety of scientific and engineering application to solve various functional equations In this method ,the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. Using the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a small parameter. Considerable research work has recently been conducted in applying this method to a class of linear and non-linear

equations. This method was further developed and improved by He, and applied to non-linear wave equations [11], boundary value problem [12].
In this paper, we further apply the Homotopy perturbation method with modified Riemann –Liouville to solve fractional- time partial differential equations, we give some examples to demonstrate the efficiency and effectiveness of the proposed method.

2-Modified Riemann-Liouville derivative:

Assume $f : R \to R, x \to f(x)$ denote a continuous (but not necessarily differentiable) function and let the partition $h > 0$ in the interval [0, 1]. Through the fractional Riemann Liouville integral

$$ {}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi , \quad \alpha > 0 \tag{2.1} $$

The modified Riemann-Liouville derivative is defined as

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-\xi)^{n-\alpha} (f(\xi) - f(0))\, d\xi, \tag{2.2} $$

Where $x \in [0,1], n-1 \le \alpha < n$ and $n \ge 1$

G.jumarie' s derivative is defined through the fractional difference

$$ \Delta^{\alpha} = (FW-1)^{\alpha} f(x) = \sum_0^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h], \tag{2.3} $$

Where $FWf(x) = f(x+h)$. Then the fractional derivative [9] is defined as the following limit,

$$ f^{(\alpha)} = \lim_{h\to 0} \frac{\Delta^{\alpha} f(x)}{h^{\alpha}} \tag{2.4} $$

The proposed modified Riemann –Liouville derivative as shown in equation. (2.2) is strictly equivalent to equation. (2.4). Meanwhile, we would introduce some properties of the fractional modified Riemann – Liouville derivative in equations. (2.5) and (2.6).

(a) Fractional Leibniz product law

$$ {}_0D_x^{\alpha}(uv) = u^{(\alpha)}v + uv^{(\alpha)} \tag{2.5} $$

(a) Fractional Leibniz formulation

$$ {}_0I_x^{\alpha}\, {}_0D_x^{\alpha} f(x) = f(x) - f(0), \quad 0 < \alpha \le 1 \tag{2.6} $$

Therefore, the integration by part can be used during the fractional calculus

$$ {}_aI_b^{\alpha} u^{(\alpha)} v = (uv)/_a^b - {}_aI_b^{\alpha} uv^{(\alpha)} \tag{2.7} $$

(b) Integration with respect to $(d\xi)^{\alpha}$.

Assume $f(x)$ denote a continuous $R \to R$ function, we use the following quality for the integral with respect to $(d\xi)^{\alpha}$

$$
\begin{aligned}
{}_0I_x^{\alpha} f(x) &= \frac{1}{\Gamma\alpha}\int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi, \qquad 0<\alpha\le 1 \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^x f(\xi)(d\xi)^{\alpha}
\end{aligned}
\tag{2.8}
$$

3-Homotopy perturbation method:

Consider the following nonlinear differential equation;

$$A(u) = f(r) \quad r \in \Omega \tag{3.1}$$

With boundary conditions:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \tag{3.2}$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.

The operator A can be generally divided in two parts L and N, where L is linear, and N is nonlinear, therefore equation. (3.1) can be written as,

$$L(u) + N(u) = f(r) \tag{3.3}$$

by using homotopy technique, one can construct a homotopy $v(r, p): \Omega \times [0,1] \to R$ which satisfies :

$$H(v, p) = (1-p)[L(v) - l(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \tag{3.4}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{3.5}$$

where $r \in \Omega$ and $p \in [0,1]$ is an embedding parameter, and u_0 is the initial approximation of equation.(3.3)which satisfies the boundary conditions. Hence, obviously we have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{3.6}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{3.7}$$

and the changing process of p from 0 to 1 is the same as changing $H(v, p)$ from $L(v) - L(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If, the embedding parameter p; $(0 \le p \le 1)$ is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of equation.(3.7)can be given as a power series in p , i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \qquad (3.8)$$

and setting $p = 1$ results in the approximate solution of equation.(3.3) as ;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \qquad (3.9)$$

4-Applicition:

In this section, we give some examples to demonstrate the efficiency and effectiveness of the Homotopy perturbation method with modified Riemann –Liouville

Example4.1 [13] Consider the linear fractional diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial (xu(x,t)}{\partial x} \quad 0 < \alpha \quad \le 1 \qquad (4.1)$$

subject to a initial condition
$u(x,0) = f(x) = 1$

we construct the following homotopy:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} + p(\frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial (xu(x,t)}{\partial x}) = 0 \qquad (4.2)$$

we consider u as

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots \qquad (4.3)$$

substituting (4.3) in (4.2) and equating the coefficients of like powers of p, we get following set of differential equations

$$p^0 : \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \; u(x,0) = 1$$

$$p^1 : \frac{\partial^\alpha u_1}{\partial t^\alpha} = -\frac{\partial^\alpha u_0}{\partial t^\alpha} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial (xu_0)}{\partial x}$$

$$p^2 : \frac{\partial^\alpha u_2}{\partial t^\alpha} = \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial (xu_1)}{\partial x}$$

$$p^3 : \frac{\partial^\alpha u_3}{\partial t^\alpha} = \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial (xu_2)}{\partial x}$$

$$p^4 : \frac{\partial^\alpha u_4}{\partial t^\alpha} = \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial (x u_3)}{\partial x}$$

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$$p^n : \frac{\partial^\alpha u_n}{\partial t^\alpha} = \frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{\partial (x u_{n-1})}{\partial x}$$

solving the systems accordingly, thus we obtain,

$$u_0(x,t) = 1$$

$$u_1 = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(-\frac{\partial^\alpha u_0}{\partial \xi^\alpha} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial (x u_0)}{\partial x}\right) d(\xi)^\alpha \Rightarrow u_1 = \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2 = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial (x u_1)}{\partial x}\right)(d\xi)^\alpha \Rightarrow u_2 = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$u_3 = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial (x u_2)}{\partial x}\right)(d\xi)^\alpha \Rightarrow u_3 = \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

$$u_4 = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial (x u_3)}{\partial x}\right)(d\xi)^\alpha \Rightarrow u_4 = \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

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$$u_n = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{\partial (x u_{n-1})}{\partial x}\right)(d\xi)^\alpha \Rightarrow u_n = \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$$

by setting $p = 1$ in equation.(4.3), the solution of (4.1) can be obtained as

$$u = u_0 + u_1 + u_2 + \ldots$$

the solution in a series form is given by

$$u(x,t) = 1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots$$

$$= \sum_0^\infty \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = E_\alpha(t^\alpha)$$

where $\sum_0^\infty \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = E_\alpha(t^\alpha)$ is the Mittag –Leffler function in one parameter .

we can readily check $u(x,t) = E_\alpha(t^\alpha)$ is an exact solution of equation.(4.1).

Example4.2 [5] Consider the fractional diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} \quad 0 < \alpha \quad \le 1 \qquad (4.4)$$

subject to a initial condition
$u(x,0) = f(x) = x^2$

we construct the following homotopy:

$$(1-p)\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) + p\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right) = 0$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} + p\left(\frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right) = 0 \qquad (4.5)$$

we consider u as

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \qquad (4.6)$$

substituting (4.6) in (4.5)and equating the coefficients of like powers of p, we get following set of differential equations:

$$p^0 : \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \quad u(x,0) = x^2$$

$$p^1 : \frac{\partial^\alpha u_1}{\partial t^\alpha} = -\frac{\partial^\alpha u_0}{\partial t^\alpha} + \frac{x^2}{2}\frac{\partial^2 u_0}{\partial x^2}$$

$$p^2 : \frac{\partial^\alpha u_2}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 u_1}{\partial x^2}$$

$$p^3 : \frac{\partial^\alpha u_3}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 u_2}{\partial x^2}$$

$$p^4 : \frac{\partial^\alpha u_4}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 u_3}{\partial x^2}$$

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$$p^n : \frac{\partial^\alpha u_n}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 u_{n-1}}{\partial x^2}$$

solving the systems accordingly, thus we obtain,

$$u_0(x,t) = x^2$$

$$u_1 = \frac{1}{\Gamma(1+\alpha)}\int_0^t \left(-\frac{\partial^\alpha u_0}{\partial \xi^\alpha} + \frac{x^2}{2}\frac{\partial^2 u_0}{\partial x^2}\right)(d\xi)^\alpha \Rightarrow u_1 = x^2 \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2 = \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{x^2}{2}\frac{\partial^2 u_1}{\partial x^2}(d\xi)^\alpha = x^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$u_3 = \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{x^2}{2}\frac{\partial^2 u_2}{\partial x^2}(d\xi)^\alpha = x^2 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

$$u_4 = \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{x^2}{2}\frac{\partial^2 u_3}{\partial x^2}(d\xi)^\alpha = x^2 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

$$\vdots$$

$$u_n = \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{x^2}{2}\frac{\partial^2 u_{n-1}}{\partial x^2}(d\xi)^\alpha = x^2 \frac{t^{n\alpha}}{\Gamma(1+4\alpha)}$$

the solution,

$$u(x,t) = x^2 + x^2\frac{t^\alpha}{\Gamma(1+\alpha)} + x^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + x^2\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + x^2\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots$$

$$= \sum_0^\infty x^2 \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = x^2 E_\alpha(t^\alpha)$$

we can readily check $u(x,t) = x^2 E_\alpha(t^\alpha)$ is an exact solution of equation.(4.4).

Example4.3 [14] Consider the two dimensional inhomogeneous wave equation:

$$\frac{\partial^\alpha u(x,y,t)}{\partial^\alpha t} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \quad 0 < \alpha \le 1 \qquad (4.7)$$

subject to a initial condition
$u(x,y,0) = \sin x \sin y$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) + p\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} + p\left(\frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) = 0$$

(4.8)

we consider u as

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \tag{4.9}$$

substituting (4.9) in (4.8) and equating the coefficients of like powers of p, we get following set of differential equations

$$p^0 : \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \quad u(x,y,0) = \sin x \sin y$$

$$p^1 : \frac{\partial^\alpha u_1}{\partial t^\alpha} = -\frac{\partial^\alpha u_0}{\partial t^\alpha} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}$$

$$p^2 : \frac{\partial^\alpha u_2}{\partial t^\alpha} = \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}$$

$$p^3 : \frac{\partial^\alpha u_3}{\partial t^\alpha} = \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}$$

$$p^4 : \frac{\partial^\alpha u_4}{\partial t^\alpha} = \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2}$$

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$$p^n : \frac{\partial^\alpha u_n}{\partial t^\alpha} = \frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{\partial^2 u_{n-1}}{\partial y^2}$$

solving the systems accordingly, thus we obtain,

$$u_0(x,y,t) = \sin x \sin y \quad u_1 = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(-\frac{\partial^\alpha u_0}{\partial \xi^\alpha} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right)(d\xi)^\alpha \Rightarrow u_1 = -2 \quad \sin x \ \sin y \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2 = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right)(d\xi)^\alpha \Rightarrow u_2 = 2^2 \sin x \ \sin y \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$u_3 = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right)(d\xi)^\alpha \Rightarrow u_3 = -2^3 \sin x \ \sin y \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

$$u_4 = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2}\right)(d\xi)^\alpha \Rightarrow u_4 = 2^4 \sin x \ \sin y \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

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$$u_n = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{\partial^2 u_{n-1}}{\partial y^2}\right)(d\xi)^\alpha \Rightarrow u_n = (-1)^n 2^n \sin x \ \sin y \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$$

the solution is

$$u(x,y,t)=\sin x\sin y-2\quad \sin x\ \sin y\frac{t^{\alpha}}{\Gamma(1+\alpha)}+2^{2}\sin x\sin y\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}-\cdots$$

$$=\sin x\sin y\sum_{0}^{\infty}(-2)^{n}\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}=\sin x\sin y\ E_{\alpha}(-2t^{\alpha})$$

we can readily check $u(x,y,t)$ is an exact solution of equation.(4.7).

Example4.4 [13] Consider the following one- dimensional linear inhomogeneous fractional wave equation

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{\partial u(x,t)}{\partial x}=\frac{t^{1-\alpha}\sin x}{\Gamma(2-\alpha)}+t\cos x \qquad (4.10)$$

$x\in R\quad t>0,\quad \alpha>0$

subject to the initial condition

$u(x,0)=0$

we construct the following homotopy:

$$(1-p)(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-\frac{\partial^{\alpha}u_{0}}{\partial t^{\alpha}})+p(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+\frac{\partial u}{\partial x}-\frac{t^{1-\alpha}\sin x}{\Gamma(2-\alpha)}-t\cos x)=0$$

$$(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-\frac{\partial^{\alpha}u_{0}}{\partial t^{\alpha}})+p(\frac{\partial^{\alpha}u_{0}}{\partial t^{\partial}}+\frac{\partial u}{\partial x}-\frac{t^{1-\alpha}\sin x}{\Gamma(2-\alpha)}-t\cos x)=0 \qquad (4.11)$$

we consider u as

$$u=u_{0}+pu_{1}+p^{2}u_{2}+p^{3}u_{3}+\cdots \qquad (4.12)$$

substituting (4.12)in (4.11)and equating the coefficients of like powers of p, we get following set of differential equations

$$p^{0}:\frac{\partial^{\alpha}u_{0}}{\partial t^{\alpha}}-\frac{\partial^{\alpha}u_{0}}{\partial t^{\alpha}}=0,\quad u(x,0)=0$$

$$p^{1}:\frac{\partial^{\alpha}u_{1}}{\partial t^{\alpha}}=-\frac{\partial^{\alpha}u_{0}}{\partial t^{\alpha}}-\frac{\partial u_{0}}{\partial x}+\frac{t^{1-\alpha}\sin x}{\Gamma(2-\alpha)}+t\cos x$$

$$p^{2}:\frac{\partial^{\alpha}u_{2}}{\partial t^{\alpha}}=-\frac{\partial u_{1}}{\partial x}$$

$$p^{3}:\frac{\partial^{\alpha}u_{3}}{\partial t^{\alpha}}=-\frac{\partial u_{2}}{\partial x}$$

$$p^4 : \frac{\partial^\alpha u_4}{\partial t^\alpha} = -\frac{\partial u_3}{\partial x}$$

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$$p^n : \frac{\partial^\alpha u_n}{\partial t^\alpha} = -\frac{\partial u_{n-1}}{\partial x}$$

solving the systems accordingly, thus we obtain,

$$u_0(x,t) = 0$$

$$u_1 = t\sin x + \frac{t^{1+\alpha}\cos x}{\Gamma(2+\alpha)}$$

$$u_2 = -\frac{t^{1+\alpha}\cos x}{\Gamma(2+\alpha)} + \frac{t^{1+2\alpha}\sin x}{\Gamma(2+2\alpha)}$$

$$u_3 = -\frac{t^{1+2\alpha}\sin x}{\Gamma(2+2\alpha)} - \frac{t^{1+3\alpha}\cos x}{\Gamma(2+3\alpha)}$$

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therefore the solution is

$$u(x,t) = t\sin x + \frac{t^{1+\alpha}\cos x}{\Gamma(2+\alpha)} - \frac{t^{1+\alpha}\cos x}{\Gamma(2+\alpha)} + \frac{t^{1+2\alpha} sinx}{\Gamma(2+2\alpha)} - \frac{t^{1+2\alpha}\sin x}{\Gamma(2+2\alpha)} - \frac{t^{1+3\alpha}\cos x}{\Gamma(2+3\alpha)} + \cdots$$

canceling the noise terms and keeping the non-noise terms yield the exact solution of equation.(4.10).

5-conclusion:

Homotopy perturbation method has been known as a powerful tool for solving many functional equations such as ordinary, partial differential equations, integral equations and fractional partial differential equations .in this article, we have presented an new form of hmotopy perturbation method with modified Riemann –Liouville derivative. The results reveal that the method is very effective and simple.

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الخلاصة:

الهدف من هذا البحث تقديم معالجة كفوءة و فعالة لطريقة الأختلال الهوموتوبي Homotopy perturabation, ومشتقة ريمان المتطورة modified Riemann -Liouville للمعادلات التفاضلية الجزئية ذات اشتقاق كسري زمني . الاشتقاق الكسري هو من وجهة نظر Jumarie. النتائج تظهر ان الطريقة المعتزمة هي جدا كفوءة وسهلة تقود الى تقريب تقارب الحلول للمعادلات التفاضلية الجزئية الكسرية.