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Homotopy Perturbation Method for Solving Fokker-Planck Equation

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Abstract:

In this paper, linear and nonlinear Fokker-Planck equations and some similar equations by using homotopy perturbation method are solved. Some examples are solved by homotopy perturbation method to illustrate the simplicity and reliability of this method.

Introduction:

The homotopy perturbation method proposed by Ji-Huan He[7,8,9]. Many authers try to improved this method to solve various nonlinear problems [2,5, 10,11,12,13]. HPM yields a very rapid convergence of the solution series and some time one iteration leads to high accuracy of the solution.

Fokker-Planck equation (FPE), first applied to investigate the Brownian motion of particles, is now largely employed in physics, engineering, biology and chemistry. Biazar and his co-authors [6] solved linear and nonlinear Fokker-Planck equation by using variational iteration method, while Tatari and his co-authors [11] used adomian decomposition method for this equation.

In this paper, we apply the homotopy perturbation method (HPM) for solve linear and nonlinear FPEs.

1- Fokker-Planck equation:

The general form of Fokker-Planck equation (FPE) for variables x and t is as follows [6]:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]u(x,t), \qquad (1)$$

With the following initial condition:

$$u(x,0) = f(x), \qquad x \in \boldsymbol{R}.$$

Here $B(x) > 0$ is called the diffusion coefficient and $A(x) > 0$ the drift coefficient. The diffusion and drift coefficients can also be functions of x and t, i.e.

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]u(x,t), \qquad (2)$$

Eq. (1) is an equation for the motion of the concentration field $u(x,t)$. Mathematically, this equation is a linear second-order partial differential equation of parabolic type. Eq. (1) is also called forward Kolmogorov equation. The backward Kolmogorov equation is written in the following form:

$$\frac{\partial u}{\partial t} = -\left[A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2}\right]u(x,t), \tag{3}$$

A generalized form of Eq. (1) to N variables $x_1, x_2, \dots, x_N$ can be written as follows:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x)\right]u(x,t), \tag{4}$$

With the following initial condition

$$u(x,0) = f(x), \qquad x = (x_1, x_2, \dots, x_N) \in \boldsymbol{R}^N.$$

Generally in Eq.(4) drift vector A_i and diffusion tensor $B_{i,j}$ depend on N variables $x_1, x_2, \dots, x_N$.

There is a more general form of FPE, which is nonlinear FPE. Nonlinear FPE has important applications in various areas such as plasma physics, surface physics, population dynamic, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing. The nonlinear FPE for one variable is in the following form:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]u(x,t). \tag{5}$$

Eq. (5) for N variables $x_1, x_2, \dots, x_N$ is in the following form:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x,t,u) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x,t,u)\right]u(x,t), \qquad x = (x_1, x_2, \dots, x_N) \in \boldsymbol{R}^N. \tag{6}$$

The paper is organized as follows: in the next section, the Homotopy perturbation method is introduced. The application of Homotopy perturbation method for solving Fokker-Planck equation is introduced in section 3. The application of the problem is obtained in section 4. In section 5, six examples explain the application. Section 6 ends this paper in conclusion.

2-Homotopy perturbation method:

To illustrate the HPM, Ji-Huan He considered the following nonlinear differential equation [5, 8]:

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{7}$$

With boundary conditions

$$B\left(u, \partial u/\partial n = 0, \qquad r \in \Gamma\right) \tag{8}$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is a boundary of the domain Ω .

The operator A can be generally divided in to two parts L and N ,where L is linear, and N is nonlinear, therefore equation(7) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{9}$$

The homotopy technique which is constructed by He [7], $v(r,p): \Omega \times [0,1] \to \mathbb{R}$ satisfied:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{10a}$$

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{10b}$$

Where $r \in \Omega$ and $p \in [0,1]$ that is called homotopy parameter, and u_0 is an initial approximation of (9), which is satisfies the boundary conditions. Obviously, from equation (10), we have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{11}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{12}$$

and the changing process of p from 0 to 1, is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic.

The embedding parameter $p \in [0,1]$ as a "small parameter" is used and assume that the solution of equation (9) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{13}$$

Setting $p = 1$ results in the approximate solution of equation (7):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{14}$$

The series (14) is convergent for most cases; however, the convergent rate depends upon the nonlinear operator $A(v)$ (the following opinions are suggested by He [7])

(1)The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e., $p \to 1$.

(2)The norm of $L^{-1}\left(\partial N/\partial v\right)$ must be smaller than one so that the series converges.

Theorem [5]

Suppose that X and Y be Banach space and $N: X \to Y$ is a contraction nonlinear mapping, that is

$$\forall\, v, \tilde{v} \in X: \quad \|N(v) - N(\tilde{v})\| \le \gamma \|v - \tilde{v}\|, \qquad 0 < \gamma < 1$$

Which according to Banach's fixed point theorem, having the fixed point u , that is $N(u) = u$.

The sequence generated by the homotopy perturbation method will be regarded as

$$V_n = N(V_{n-1}), \qquad V_{n-1} = \sum_{i=0}^{n-1} u_i, \qquad n = 1,2,3 \ldots .$$

and suppose that $V_0 = v_0 = u_0 \in B_r(u)$ where $B_r(u) = \left\{u^* \in \frac{X}{\|u^* - u\|} < r\right\}$, then we have the following statements:

(i) $\|V_n - u\| \le \gamma^n \| v_0 - u\|$.

(ii) $V_n \in B_r(u)$.

(iii) $\lim_{n\to\infty} V_n = u$.

Proof: (i) By the induction method on n , for $n = 1$ we have

$$\|V_1 - u\| = \|N(V_0) - N(u)\| \le \gamma\|v_0 - u\|.$$

Assume that $\|V_{n-1} - u\| \le \gamma^{n-1}\|v_0 - u\|$ as an induction hypothesis, then

$$\|V_n - u\| = \|N(V_{n-1}) - N(u)\| \le \gamma\|V_{n-1} - u\| \le \gamma\gamma^{n-1}\|v_0 - u\| = \gamma^n\|v_0 - u\|.$$

(ii) Using (i), we have

$$\|V_n - u\| \le \gamma^n\|v_0 - u\| \le \gamma^n r < r \Rightarrow V_n \in B_r(u).$$

(iii) Because of $\|V_n - u\| \le \gamma^n\|v_0 - u\|$, and $\lim_{n\to\infty}\gamma^n = 0$, we drive

$\lim_{n\to\infty}\|V_n - u\| = 0$, that is $\lim_{n\to\infty} V_n = u$.

3- Application of homotopy perturbation method:

1- At first, we construct a homotopy perturbation method for equation (1) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v\right) = 0 \quad (15)$$

or

$$\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u_0}{\partial t} - \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v\right) = 0 \quad (16)$$

By substituting (13) into (16) and equating the coefficients of like terms with the identical powers of p , we obtain:

$$p^0: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \quad (17)$$

$$p^1: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} + \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v_0$$

$$= -\frac{\partial u_0}{\partial t} + \left(-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right)v_0 + \left(-A(x) + 2\frac{\partial B(x)}{\partial x}\right)\frac{\partial v_0}{\partial x} + B(x)\frac{\partial^2 v_0}{\partial x^2} \quad (18)$$

$$p^2: \frac{\partial v_2}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v_1$$

$$= \left(-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right) v_1 + \left(-A(x) + 2\frac{\partial B(x)}{\partial x}\right)\frac{\partial v_1}{\partial x} + B(x)\frac{\partial^2 v_1}{\partial x^2} \qquad (19)$$

⋮

$$p^k: \frac{\partial v_k}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right] v_{k-1}$$

$$= \left(-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right) v_{k-1} + \left(-A(x) + 2\frac{\partial B(x)}{\partial x}\right)\frac{\partial v_{k-1}}{\partial x} + B(x)\frac{\partial^2 v_{k-1}}{\partial x^2} \qquad (20)$$

By integration of the both sides of the equations, we obtain the following multiple solutions:

$$v_0 = u_0 \qquad (21)$$

$$v_k = \int_0^t \left[\left(-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right) v_{k-1} + \left(-A(x) + 2\frac{\partial B(x)}{\partial x}\right)\frac{\partial v_{k-1}}{\partial x} + B(x)\frac{\partial^2 v_{k-1}}{\partial x^2}\right] \mathrm{dt}$$

$$k = 1,2,3,\ldots \qquad (22)$$

2- We construct a homotopy perturbation method for equation (2) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right] v\right) = 0 \qquad (23)$$

Or

$$\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u_0}{\partial t} - \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right] v\right) = 0 \qquad (24)$$

By substituting (13) into (24) and equating the coefficients of like terms with the identical powers of p , we obtain:

$$p^0: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \qquad (25)$$

$$p^1: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} + \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right] v_0$$

$$= \left(-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2}\right) v_0 + \left(-A(x,t) + 2\frac{\partial B(x,t)}{\partial x}\right)\frac{\partial v_0}{\partial x} + B(x,t)\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial u_0}{\partial t} \qquad (26)$$

$$p^2: \frac{\partial v_2}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right] v_1$$

$$= \left(-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2}\right) v_1 + \left(-A(x,t) + 2\frac{\partial B(x,t)}{\partial x}\right)\frac{\partial v_1}{\partial x} + B(x,t)\frac{\partial^2 v_1}{\partial x^2} \qquad (27)$$

⋮

$$p^k: \frac{\partial v_k}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t) + \frac{\partial^2}{\partial x^2} B(x,t) \right] v_{k-1}$$

$$= \left(-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2} \right) v_{k-1} + \left(-A(x,t) + 2\frac{\partial B(x,t)}{\partial x} \right) \frac{\partial v_{k-1}}{\partial x} + B(x,t) \frac{\partial^2 v_{k-1}}{\partial x^2} \quad (28)$$

By integration of the both sides of the equations, we obtain the following multiple solutions:

$$v_0 = u_0 \quad (29)$$

$$v_k = \int_0^t \left[\left(-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2} \right) v_{k-1} + \left(-A(x,t) + 2\frac{\partial B(x,t)}{\partial x} \right) \frac{\partial v_{k-1}}{\partial x} + B(x,t) \frac{\partial^2 v_{k-1}}{\partial x^2} \right] \mathbf{dt}$$

$$k = 1,2,3,\ldots \quad (30)$$

3- We construct a homotopy perturbation method for equation (3) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} \right) + p\left(\frac{\partial v}{\partial t} + \left[A(x)\frac{\partial}{\partial x} + B(x)\frac{\partial^2}{\partial x^2} \right] v \right) = 0 \quad (31)$$

or

$$\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} \right) + p\left(\frac{\partial u_0}{\partial t} + \left[A(x)\frac{\partial}{\partial x} + B(x)\frac{\partial^2}{\partial x^2} \right] v \right) = 0 \quad (32)$$

By substituting (13) into (32) and equating the coefficients of like terms with the identical powers of p , we obtain:

$$p^0: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \quad (33)$$

$$p^1: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} - \left[A(x)\frac{\partial v_0}{\partial x} + B(x)\frac{\partial^2 v_0}{\partial x^2} \right] \quad (34)$$

$$p^2: \frac{\partial v_2}{\partial t} = -\left[A(x)\frac{\partial v_1}{\partial x} + B(x)\frac{\partial^2 v_1}{\partial x^2} \right] \quad (35)$$

$\vdots$

$$p^k: \frac{\partial v_k}{\partial t} = -\left[A(x)\frac{\partial v_{k-1}}{\partial x} + B(x)\frac{\partial^2 v_{k-1}}{\partial x^2} \right] \quad (36)$$

By integration of the both sides of the equations, we obtain the following multiple solutions:

$$v_0 = u_0 \quad (37)$$

$$v_k = -\int_0^t \left[A(x)\frac{\partial v_{k-1}}{\partial x} + B(x)\frac{\partial^2 v_{k-1}}{\partial x^2} \right] \mathbf{dt}$$

$$k = 1,2,3,\dots \tag{38}$$

4- We construct a homotopy perturbation method for equation (4) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial v}{\partial t}-\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x)+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(x)\right]v\right)=0 \tag{39}$$

or

$$\left(\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial u_0}{\partial t}-\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x)+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(x)\right]v\right)=0 \tag{40}$$

By substituting (13) into (40) and equating the coefficients of like terms with the identical powers of p , we obtain:

$$p^0:\frac{\partial v_0}{\partial t}=\frac{\partial u_0}{\partial t} \tag{41}$$

$$p^1:\frac{\partial v_1}{\partial t}=-\frac{\partial u_0}{\partial t}+\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x)+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(x)\right]v_0$$

$$=-\frac{\partial u_0}{\partial t}-v_0\sum_{i=1}^{N}\frac{\partial A_i(x)}{\partial x_i}+v_0\sum_{i,j=1}^{N}\frac{\partial^2 B_{i,j}(x)}{\partial x_i\partial x_j}-\sum_{i=1}^{N}A_i(x)\frac{\partial v_0}{\partial x_i}+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x)}{\partial x_i}\frac{\partial v_0}{\partial x_j}$$
$$+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x)}{\partial x_j}\frac{\partial v_0}{\partial x_i}+\sum_{i,j=1}^{N}B_{i,j}(x)\frac{\partial^2 v_0}{\partial x_i\partial x_j} \tag{42}$$

$\vdots$

$$p^k:\frac{\partial v_k}{\partial t}=-\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x)+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(x)\right]v_{k-1}$$

$$=-v_{k-1}\sum_{i=1}^{N}\frac{\partial A_i(x)}{\partial x_i}+v_{k-1}\sum_{i,j=1}^{N}\frac{\partial^2 B_{i,j}(x)}{\partial x_i\partial x_j}-\sum_{i=1}^{N}A_i(x)\frac{\partial v_{k-1}}{\partial x_i}+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x)}{\partial x_i}\frac{\partial v_{k-1}}{\partial x_j}$$
$$+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x)}{\partial x_j}\frac{\partial v_{k-1}}{\partial x_i}+\sum_{i,j=1}^{N}B_{i,j}(x)\frac{\partial^2 v_{k-1}}{\partial x_i\partial x_j} \tag{43}$$

By integration of the both sides of the equations, we obtain the following multiple solutions:

$$v_0=u_0 \tag{44}$$

$$v_k=-\int_0^t\left[\begin{gathered}-v_{k-1}\sum_{i=1}^{N}\frac{\partial A_i(x)}{\partial x_i}+v_{k-1}\sum_{i,j=1}^{N}\frac{\partial^2 B_{i,j}(x)}{\partial x_i\partial x_j}-\sum_{i=1}^{N}A_i(x)\frac{\partial v_{k-1}}{\partial x_i}+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x)}{\partial x_i}\frac{\partial v_{k-1}}{\partial x_j}\\+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x)}{\partial x_j}\frac{\partial v_{k-1}}{\partial x_i}+\sum_{i,j=1}^{N}B_{i,j}(x)\frac{\partial^2 v_{k-1}}{\partial x_i\partial x_j}\end{gathered}\right]dt$$

$k = 1,2,3, \ldots$ **(45)**

5- We construct a homotopy perturbation method for equation (5) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]v\right) = 0 \qquad (46)$$

or

$$\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u_0}{\partial t} - \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]v\right) = 0 \qquad (47)$$

By substituting (13) into (47) and equating the coefficients of like terms with the identical powers of p , we obtain:

$$p^0: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \qquad (48)$$

$$p^1: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} + \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]v_0$$

$$= \left(-\frac{\partial A(x,t),u}{\partial x} + \frac{\partial^2 B(x,t,u)}{\partial x^2}\right)v_1 + \left(-A(x,t,u) + 2\frac{\partial B(x,t,u)}{\partial x}\right)\frac{\partial v_1}{\partial x} + B(x,t,u)\frac{\partial^2 v_1}{\partial x^2} \qquad (49)$$

$$p^2: \frac{\partial v_2}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]v_1$$

$$= \left(-\frac{\partial A(x,t),u}{\partial x} + \frac{\partial^2 B(x,t,u)}{\partial x^2}\right)v_1 + \left(-A(x,t,u) + 2\frac{\partial B(x,t,u)}{\partial x}\right)\frac{\partial v_1}{\partial x} + B(x,t,u)\frac{\partial^2 v_1}{\partial x^2} \qquad (50)$$

$\vdots$

$$p^k: \frac{\partial v_k}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]v_{k-1}$$

$$= \left(-\frac{\partial A(x,t,u)}{\partial x} + \frac{\partial^2 B(x,t,u)}{\partial x^2}\right)v_{k-1} + \left(-A(x,t,u) + 2\frac{\partial B(x,t,u)}{\partial x}\right)\frac{\partial v_{k-1}}{\partial x} + B(x,t,u)\frac{\partial^2 v_{k-1}}{\partial x^2} \qquad (51)$$

By integration of the both sides of the equations, we obtain the following multiple solutions:

$$v_0 = u_0 \qquad (52)$$

$$v_k = \int_0^t \left[\left(-\frac{\partial A(x,t,u)}{\partial x} + \frac{\partial^2 B(x,t,u)}{\partial x^2}\right)v_{k-1} + \left(-A(x,t,u) + 2\frac{\partial B(x,t,u)}{\partial x}\right)\frac{\partial v_{k-1}}{\partial x} + B(x,t,u)\frac{\partial^2 v_{k-1}}{\partial x^2}\right]\mathrm{dt}$$

$k = 1,2,3, \ldots$ (53)

6- We construct a homotopy perturbation method for equation (6) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial v}{\partial t}-\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x,t,v)+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(x,t,v)\right]v\right)=0 \qquad (54)$$

or

$$\left(\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial u_0}{\partial t}-\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x,t,v)+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(x,t,v)\right]v\right)=0 \qquad (55)$$

By substituting (13) into (40) and equating the coefficients of like terms with the identical powers of p , we obtain:

$$p^0:\frac{\partial v_0}{\partial t}=\frac{\partial u_0}{\partial t} \qquad (56)$$

$$p^1:\frac{\partial v_1}{\partial t}=-\frac{\partial u_0}{\partial t}+\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x,t,v_0)+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(x,t,v_0)\right]v_0$$

$$=-\frac{\partial u_0}{\partial t}-v_0\sum_{i=1}^{N}\frac{\partial A_i(x,t,v_0)}{\partial x_i}+v_0\sum_{i,j-1}^{N}\frac{\partial^2 B_{i,j}(x,t,v_0)}{\partial x_i\partial x_j}-\sum_{i=1}^{N}A_i(x,t,v_0)\frac{\partial v_0}{\partial x_i}+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x,t,v_0)}{\partial x_i}\frac{\partial v_0}{\partial x_j}$$

$$+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x,t,v_0)}{\partial x_j}\frac{\partial v_0}{\partial x_i}+\sum_{i,j=1}^{N}B_{i,j}(x,t,v_0)\frac{\partial^2 v_0}{\partial x_i\partial x_j} \qquad (57)$$

⋮

$$p^k:\frac{\partial v_k}{\partial t}=-\left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x,t,v_{k-1})+\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}B_{i,j}(x,t,v_{k-1})\right]v_{k-1}$$

$$=-v_{k-1}\sum_{i=1}^{N}\frac{\partial A_i(x,t,v_{k-1})}{\partial x_i}+v_{k-1}\sum_{i,j=1}^{N}\frac{\partial^2 B_{i,j}(x,t,v_{k-1})}{\partial x_i\partial x_j}-\sum_{i=1}^{N}A_i(x,t,v_{k-1})\frac{\partial v_{k-1}}{\partial x_i}+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x,t,v_{k-1})}{\partial x_i}\frac{\partial v_{k-1}}{\partial x_j}$$

$$+\sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x,t,v_{k-1})}{\partial x_j}\frac{\partial v_{k-1}}{\partial x_i}+\sum_{i,j=1}^{N}B_{i,j}(x,t,v_{k-1})\frac{\partial^2 v_{k-1}}{\partial x_i\partial x_j} \qquad (58)$$

By integration of the both sides of the equations, we obtain the following multiple solutions:

$$v_0=u_0 \qquad (59)$$

$$v_k = -\int_0^t \left[\begin{array}{l} -v_{k-1}\sum_{i=1}^{N}\frac{\partial A_i(x,t,v_{k-1})}{\partial x_i} + v_{k-1}\sum_{i,j=1}^{N}\frac{\partial^2 B_{i,j}(x,t,v_{k-1})}{\partial x_i \partial x_j} - \sum_{i=1}^{N} A_i(x,t,v_{k-1})\frac{\partial v_{k-1}}{\partial x_i} + \sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x,t,v_{k-1})}{\partial x_i}\frac{\partial v_{k-1}}{\partial x_j} \\ + \sum_{i,j=1}^{N}\frac{\partial B_{i,j}(x,t,v_{k-1})}{\partial x_j}\frac{\partial v_{k-1}}{\partial x_i} + \sum_{i,j=1}^{N} B_{i,j}(x,t,v_{k-1})\frac{\partial^2 v_{k-1}}{\partial x_i \partial x_j} \end{array} \right] dt$$

$$k = 1,2,3,\dots \qquad (60)$$

Such that $u(x,0) = u_0(x,t)$, so $\frac{\partial u_0}{\partial t} = 0$.

The approximate solution is:

$$u = v_0 + v_1 + v_2 + \cdots \qquad (61)$$

4-Examples:

Example 1

Consider Eq.(1) with the following initial condition:

$$u(x,0) = x, \qquad x \in R \qquad (62)$$

Let in Eq.(1)

$$A(x) = -1, \qquad (63)$$

$$B(x) = 1. \qquad (64)$$

Assuming $u_0(x,t) = x$, as an initial approximation that satisfies the initial condition, from Eq.(21) and substituting equations (63) and (64) into Eq.(22) we obtain

$u_1 = t,$

$u_2 = 0,$

$u_3 = 0,$

$\vdots$

So that the solution of Eq.(1) will be as follows:

$$u(x,t) = u_0 + u_1 = x + t$$

Example 2

In this example we consider Eq.(2) with the initial condition:

$$u(x,0) = \sinh(x), \qquad x \in R \qquad (67)$$

Let the drift and diffusion coefficient in Eq.(2) be in the following form:

$$A(x,t) = e^t(\coth(x)\cosh(x) + \sinh(x)) - \sinh(x), \qquad (68)$$

$$B(x,t) = e^t\cosh(x). \qquad (69)$$

Selecting $u_0(x,t) = \sinh(x)$, as an initial approximation in Eq.(29), and substituting equations (68) and (69) into Eq.(30) we obtain the following successive approximations:

$u_1 = t\sinh(x),$

$u_2 = \frac{t^2}{2}\sinh(x),$

$u_3 = \frac{t^3}{6}\sinh(x),$

$u_4 = \frac{t^4}{24}\sinh(x),$

$\vdots$

Therefore,

$$u(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\sinh(x)$$

That leads to the following solution:

$u = e^t \sinh(x)$.

Example 3

Consider the backward Kolmogorov Eq.(3) and let the initial condition be given by

$$u(x,0) = x + 1, \qquad x \in \boldsymbol{R} \tag{70}$$

Also, we consider

$$A(x,t) = -(x+1), \tag{71}$$

$$B(x,t) = x^2 e^t. \tag{72}$$

Assuming $u_0(x,t) = \sinh(x)$ in Eq.(37) and substituting equations (71) and (72) into Eq.(38) we obtain the following successive approximations:

$u_1 = t(x+1),$

$u_2 = \frac{t^2}{2(x+1)},$

$u_3 = \frac{t^3}{6(x+1)},$

$u_4 = \frac{t^4}{24(x+1)},$

$\vdots$

Thus, we obtain

$$u(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)(x+1)$$

which is equivalent to the following closed form of the solution:
$u(x,t) = e^t(x+1)$.

Example 4

Consider Eq.(4), with the initial condition

$$u(x,0) = x_1, \qquad x = (x_1, x_2)^T \in \mathbf{R}^2 \tag{73}$$

Also let

$$\begin{cases} A_1(x_1,x_2) = x_1, \\ A_2(x_1,x_2) = 5x_2, \end{cases} \tag{74}$$

$$\begin{cases} B_{1,1}(x_1,x_2) = x_1^2, \\ B_{1,2}(x_1,x_2) = 1, \\ B_{2,1}(x_1,x_2) = 1, \\ B_{2,2}(x_1,x_2) = x_2^2, \end{cases} \tag{75}$$

Consider $u_0(x_1,x_2,t) = x_1$ as the zeroth approximation, using this selection in Eq.(44) and substituting equations (71) and (72) into Eq.(45) we obtain the following successive approximation:

$$u_1 = x_1 t,$$

$$u_2 = x_1 \frac{t^2}{2},$$

$$u_3 = x_1 \frac{t^3}{6},$$

$$u_4 = x_1 \frac{t^4}{24},$$

$$\vdots$$

Thus, the solution of Eq.(4) will be as the follows:

$$u(x_1,x_2,t) = x_1\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)$$

which is equivalent to the following closed form of the solution:
$u(x_1,x_2,t) = x_1 e^t$.

Example 5

Consider the nonlinear FPE (5) such that

$$u(x,0) = x^2, \qquad x \in \mathbf{R}^2 \tag{76}$$

$$A(x,t,u) = \frac{4}{x}u - \frac{x}{3}, \tag{77}$$

$$B(x,t,u) = u. \tag{78}$$

Substituting these values in Eq.(53) and considering $u_0(x,t) = x^2$ by the Eq.(52) we have

$$u_1 = x^2 t,$$

$u_2 = x^2 \frac{t^2}{2}$,

$u_3 = x^2 \frac{t^3}{6}$,

$u_4 = x^2 \frac{t^4}{24}$,

$\vdots$

Thus, the solution will be as follows:

$u(x,t) = x^2 e^t$.

Example 6

Consider the generlized nonlinear Eq.(6), with the initial condition

$$u(x,0) = x_1^2, \qquad x = (x_1, x_2)^T \in \boldsymbol{R}^2 \tag{79}$$

Also let

$$\begin{cases} A_1(x_1,x_2) = \frac{4}{x_1} u, \\ A_2(x_1,x_2) = x_2, \end{cases} \tag{80}$$

$$\begin{cases} B_{1,1}(x_1,x_2) = u, \\ B_{1,2}(x_1,x_2) = 1, \\ B_{2,1}(x_1,x_2) = 1, \\ B_{2,2}(x_1,x_2) = u, \end{cases} \tag{81}$$

By substitute these equations in Eq.(60) and selecting $u_0(x_1,x_2,t) = x_1^2$, from Eq.(59), we drive the following results:

$u_1 = -x_1^2 t$,

$u_2 = x_1^2 \frac{t^2}{2}$,

$u_3 = -x_1^2 \frac{t^3}{6}$,

$u_4 = x_1^2 \frac{t^4}{24}$,

$\vdots$

Therefore, the solution of Eq.(6) is in the following form:

$$u(x,t) = x_1^2 \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} = x_1^2 e^{-t}.$$

The solutions obtained in examples (1-5) are the same those obtained by ADM [11] and VIM [6], and the solution obtained in example (6) are the same this obtained by VIM [6].

Conclusion:

We solved the Fokker-Planck equation by homotopy perturbation method. We notice from the examples that the HPM is very accurate method since the results of this method are the same results of

ADM and VIM. So that the HPM is remarkably effective for solve the Fokker-Planck equation. In our work, we use the Maple13 to calculate the results which are obtained from the iteration method HPM.

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الملخص:

في هذا البحث, سنقوم بحل معادلة فوكر–بلانك وبعض المعادلات المشابهة لها باستخدام طريقة اضطراب الهوموتوبي. بعض الامثلة حلت باستخدام طريقة اضطراب الهوموتوبي لبيان بساطة وقابلية تلك الطريقة .