J.Thi-Qar Sci. Vol.2 (2) April/2010

ISSN 1991- 8690 الترقيم الدولي ١٩٩١ - ٨٦٩٠

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EXTENSIBILITY OF GRAPHS

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Abstract

In this paper, the concepts of *extension of a graph*(*digraph*) and the *extensible class* of graphs(digraphs) have been introduced. The class of connected graphs as well as the class of Hamiltonian graphs which are extensible classes have also been proved. The classes of regular, eulerian, bipartite and trees graphs which are not extensible classes have also been proved.

The concept of *extensibility number* has been introduced as well as the characterization of regular graphs(digraphs) which have extensibility number k. Also the extensibility number of eulerian graphs(digraphs) has been characterized.

Key words: Joining graphs, Extension of graph**s,** Regular graphs, Reducibility, Contractibility, and Connectivity.

1. Introduction

Kharat and **Waphare** [**2001**] introduced the concept of reducibility number for posts in Lattices Theory. **Akram** [**2005**] introduced analogous concept in graph theory. In fact, he studied the reducibility number for some classes of graphs. **Akram** [**2007**] introduced the contractibility number for some classes of graphs. In this work, we introduced the concept of extensibility number in graphs.

A *graph* $G=(V(G),E(G))$ consists of two finite sets, $V(G)$, the *vertex set* of the graph, often denoted by just V, which is a nonempty set of elements called *vertices*, and E(G), the *edge set* of the graph, often denoted by just E, which is a possibly empty set of elements called *edges*, such that each edge e in E is assigned an unordered pair of vertices (u,v) called the *end vertices* of e.The number of vertices of G will be called the *order* of G, and will usually be denoted by p; the number of edges of G will generally be denoted by q. If for a graph G, $p=1$ then G is called *trivial graph*; if $q=0$ then G is called a *null graph*. We shall usually denote the edge corresponding to (v,w) where (v and w are vertices of G) by vw.

If e is an edge of G having end vertices v,w then e is said to *join* the vertices v and w, and these vertices are then said to be *adjacent*. In this case, we also say that e is *incident* to v and w, and that w is a *neighbor* of v. An *independent set of vertices* in G is a set of vertices of G no two of which are adjacent.

Let v be a vertex of the graph G. If v joined to itself by an edge, such an edge is called *loop*. The degree $d(v)$ is the number of edges of G incident with v, counting each loop twice. If two (or more) edges of G have the same end vertices then these edges are called *parallel*. A graph is called *simple* if it has no loops and parallel edges. We say that G is $r-$regular graph if the degree of every vertex is r. A simple graph in which every two vertices are adjacent is called a *complete graph*; the complete graph with p vertices is denoted by K_p. A *bipartite graph* is one whose vertex set can be partitioned into two subsets V_1 and V_2 in such away that each edge joins a vertex of V_1 to a vertex of V_2.

A *walk* in a graph G is a finite sequence $W = v_0e_1v_1\, e_2 ... v_{k-1}e_kv_k$ whose terms are alternatively vertices and edges such that for $1 \le i \le k$, the edge e_i has ends v_{i-1} and v_i. The vertex v_0 is called the *origin* of the walk W, while v_k is called the *terminus* of W. The vertices $v_1,...,v_{k-1}$ in the above walk W are called *internal vertices*. If the edges $e_1, e_2,...,e_k$ of the walk $W = v_0e_1v_1\, e_2 ... v_{k-1}e_kv_k$ are distinct then W is called a *trail* and if $v_0 = v_k$ then W is called a *closed trail*. If the vertices $v_0, v_1,...,v_k$ of the walk $W = v_0e_1v_1\, e_2 ... v_{k-1}e_kv_k$ are distinct then W is called a *path*. A path with n vertices will sometime be denoted by P_n. A closed trail in a graph G is called a *cycle* if its origin and internal vertices are distinct. A cycle with n vertices, will sometime be denoted by C_n and called n-cycle. A trail in G is called *Euler trail* if it includes every edge of G. A *tour* of G is a closed walk of G which includes every edge of G at least once. An *Euler tour* of G is a tour which includes each edge exactly once. A graph G is called Eulerian or Euler if it has an Euler tour.

A graph G is *connected* if there is a path joining each pair of vertices of G; a graph which is not connected is called *disconnected*. A connected graph which contains no cycle is called a *tree*. A graph G is *Hamiltonian* if it has a cycle which includes every vertex of G. The *vertex connectivity* of G, denoted $\kappa(G)$ is the smallest number of vertices in G whose deletion from G leaves either a disconnected graph or K_1. A simple graph G is called n-*connected* (where $n \ge 1$) if $\kappa(G) \ge n$.

A *directed graph* $D = (V, A)$ consists of two finite sets V, the vertex set, a nonempty set of elements called the *vertices* of D and A, the arc set, a (possible empty) set of elements called the *arcs* of D, such that each arc a in A is assigned an ordered pair of vertices (u, v).

If a is an arc, in the directed graph D, with associated ordered pair of vertices (u, v), then a is said to *join* u to v, u is called the *origin* or the *initial vertex* or the *tail* of a, and v is called the *terminus* or the *terminal vertex* or the *head* of a.

Given a digraph D we can obtain a graph G from D by "removing all the arrows" from the arcs. G is then called the *underlying graph* of D.

Let D be a digraph. Then the *directed walk* in D is a finite sequence $W = v_0a_1v_1...a_kv_k$, whose terms are alternately vertices and arcs such that for $i = 1,2,...,k$, the arc a_i has origin v_{i-1} and terminus v_i.

There are similar definitions for *directed trails, directed paths, directed cycles* and *directed tours*.

A vertex v of the digraph D is said to *reachable* from a vertex u if there is a directed path in D from u to v. A digraph

D is said to be *connected* if its underlying graph is connected. A digraph D is called *simple* if, for any pair of vertices u and v of D, there is at most one arc from u to v and there is no arc from u to itself.

Let v be a vertex in the digraph D. The *indegree* $id(v)$ of v is the number of arcs of D that have v as its head, i.e., the number of arcs that "go to" v. Similarly, the *outdegree* $od(v)$ of v is the number of arcs of D that have v as its tail, i.e., that "go out" of v.

Let D be a connected digraph. Then a *directed Euler trail* in D is a directed open trail of D containing all the arcs of D (once and only once). *A directed Euler tour* of D is a directed closed trail of D containing all the arcs of D (once and only once). A digraph D containing a directed Euler tour is called an *Euler digraph*. A digraph D is called $k-$*regular* if $id(v) = od(v) = k$ for each vertex v of D. A *directed Hamiltonian cycle* in a digraph D is a directed cycle which includes every vertex of D. If D contains such a cycle then D is called *Hamiltonian*.

For the undefined concepts and terminology we refer the reader to Wilson[1978], Clark[1991], Harary[1969], West[1999] and Tutte[1984].

2. Extensibility of Graphs.

In this section, we introduced the concepts of extension of graphs, extensible class of graphs and the extensibility number of graph. Further, we characterized the extensibility number of regular and eulerian graphs.

Definition 2.1(Clark[1991]): Let G_1 and G_2 be two graphs with no vertex in common. We define the *join* of G_1 and G_2 denoted by $G_1 + G_2$ to be the graph with vertex set and edge set given as follows:

$$V(G_1 + G_2) = V(G_1) \cup V(G_2),$$

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$$

Where $J = \{x_1 x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$. Thus J consists of edges which join every vertex of G_1 to every vertex of G_2.

Here we introduce the concept of extension of a graph.

Definition 2.2: Let G be a nontrivial graph. The *extension* of G is a graph denoted by $G + S$ obtained from G by adding a nonempty set of independent vertices S such that every vertex in S is adjacent to every vertex in G exactly one. In such away S is called *extension set* of G. In particular, if S consists of a single element v, then v is called *extension vertex* of G. The graph $G + S$ have vertex set and edge set as follows:

$$V(G + S) = V(G) \cup S,$$

$$E(G + S) = E(G) \cup J$$

Where $J = \{x_1 x_2 : x_1 \in S, x_2 \in V(G)\}$. Thus J consists of edges which join every vertex of S to every vertex of G.

Definition 2.3: Let $\Im$ be the class of graphs with certain property. Then $\Im$ is called *extensible class*, if for every graph $G \in \Im$, there exists an extension vertex v of G such that $G + v \in \Im$.

Proposition 2.4:
1. the class of connected graphs is extensible class.
2. the class of Hamiltonian graphs is extensible class.

Proof: (1) The proof follows from definition 2.2.

(2) Let G be any Hamiltonian graph with n vertices, and $C = v_1 v_2 ... v_n v_1$ be the Hamiltonian cycle of G. Suppose that v_0 is extension vertex of G. By definition 2.2, v_0 is adjacent to every vertex in G exactly one. Then the cycle $C_0 = v_0 v_1 v_2 ... v_n v_0$ is Hamiltonian in the graph $G + v_0$. Hence the graph $G + v_0$ is Hamiltonian graph and the proof of (2) follows. □

Proposition 2.5:

1. the class of trees is not extensible class.
2. the class of bipartite graphs is not extensible class.
3. the class of regular graphs is not extensible class.
4. the class of eulerian graphs is not extensible class.

Proof: (1) Let G be any tree**.** Then G is connected graph without cycle. As the extension vertex v of G is adjacent to every vertex in G. Hence the graph $G+v$ has a cycle and the proof of (1) follows.

(2) Let G be a bipartite graph. Then the vertex set V of G can be partitioned into two subsets of vertices V_1 and V_2 such that every edge in G join a vertex in V_1 to a vertex in V_2. Let v_0 be extension vertex of G. By definition 2.2, v_0 is adjacent to all the vertices in V_1 and V_2. Then v_0 with any two adjacent vertices in G form an odd cycle. Then the graph $G+v_0$ contains an odd cycle. But all the cycles in any bipartite graph is even Harary[1969]. Hence $G+v_0$ is not bipartite graph and the proof of (2) follows.

(3) Let G be an r-regular graph with n vertices which is not complete graph, and v_0 be an extension vertex of G. By definition 2.2, v_0 is adjacent to every vertex in G exactly one. Thus $d(v_0)=n$ and $d(v_i)=r+1 \quad \forall i=1,2,...,n$ in the graph $G+v_0$. As G is not complete graph, then $r \neq n-1$ which implies $n \neq r+1$. Hence the graph $G+v_0$ is not regular and the proof of (3) follows.

(4) Let G be an eulerian graph. Then G is connected and every vertex in G has even degree. By definition 2.2, the extension vertex v_0 of G increase the degree of every vertex in G by 1. Therefore, the degree of every vertex of G in the graph $G+v_0$ is odd. Hence $G+v_0$ is not eulerian graph and the proof of (4) follows. □

Now, let $\Im$ be the class of graphs with certain property, $G \in \Im$. The question is, what is the smallest positive integer m such that there exists an extension set S of G with cardinality m for which $G+S \in \Im$. In order to takle this question we define the extensibility number.

Definition 2.6: let $\Im$ be the class of graphs with certain property, and $G \in \Im$ be a nontrivial. The *extensibility number* of G with respect to $\Im$ is the smallest positive integer m, if exists, such that there exists an extension set S of G with cardinality m in which the new graph $G+S \in \Im$. We write $m = \underset{\Im}{ext}(G)$. If such a number does not exist for G, then we say the corresponding extensibility number is ∞.

One can see immediate, the class of graphs $\Im$ is extensible class if and only if the extensibility number of every graph $G \in \Im$ is one. Further, the extensibility number for each of the classes of trees and bipartite graphs is ∞.

Now, we characterize the extensibility number for regular and eulerian graphs.

Theorem 2.7: Let $\Re$ be the class of regular graphs, $R \in \Re$**.** Then $\underset{\Re}{ext}(R)=1$ if and only if R is a complete or trivial graph.

Proof: Let R be an r-regular graph with number of vertices n.

Suppose that $\underset{\Re}{ext}(R)=1$. By definition 2.6, there exists an extension set of R with single element v_0 such that $R+v_0 \in \Re$. By definition 2.2, v_0 is adjacent to every vertex in R exactly one. That is $d(v_0)=n$ and v_0 increase the degree of every vertex in R by 1. Then the degree of every vertex of R in the graph $R+v_0$ is $r+1$. As the graph $R+v_0$ is regular, we must have $n=r+1$. If $r=0$, then $n=1$ and R is a trivial graph, otherwise $r=n-1$ and R is a complete graph.

Conversely, if R is trivial graph, then it is not difficult to see that $\underset{\Re}{ext}(R)=1$. Suppose that R is a complete graph with number of vertices n. Then R is regular graph with regularity degree $n-1$. We prove that $\underset{\Re}{ext}(R)=1$. Let v_0 be a vertex different from the vertices of R and join it to every vertex in R exactly one. Then $d(v_0)=n$ and v_0 increase the degree of every vertex in R by 1. As the degree of every vertex in R is $n-1$, then the degree of every vertex in R after joining v_0 is $n-1+1=n$. Thus the new graph $R+v_0$ is n-regular graph. As such v_0 is extension vertex of R with respect to $\Re$. Hence $\underset{\Re}{ext}(R)=1$. □

Theorem 2.8: Let $\Re$ be the class of regular graphs, R be an r-regular graph with n vertices in $\Re$. Then $\underset{\Re}{ext}(R)=k$ if and only if k is the smallest number of vertices and R has regularity degree $r=n-k$.

Proof: Let R be an r-regular graph with n vertices.

Suppose that $\underset{\Re}{ext}(R)=k$. Then by definition 2.6, there exists an extension set $S=\{v_1,v_2,...,v_k\}$ of R and k is the smallest cardinality of S such that the graph $R+S\in\Re$. By definition 2.2, S is independent set of vertices and every vertex in S is adjacent to every vertex in R exactly one. Thus the degree of every vertex of S in the graph $R+S$ is n. That is $d(v_1)=n, d(v_2)=n,...,d(v_k)=n,$ and the degree of every vertex in R in the graph $R+S$ is $r+k$. As $R+S$ is regular graph, then we must have $n=r+k$ which implies $r=n-k$.

Conversely, suppose that k is the smallest number of vertices and R has regularity degree $r=n-k$. We prove that $\underset{\Re}{ext}(R)=k$. Let $S=\{v_1,v_2,...,v_k\}$ be a set of independent vertices with cardinality k and vertices different from the vertices of R. Join every vertex of S to every vertex in R exactly one. Then we get that the degree of every vertex in S in the graph $R+S$ is n. That is $d(v_1)=d(v_2)=...=d(v_k)=n$ and the degree of every vertex of R in the graph $R+S$ is $r+k$. By assumption $r=n-k$, this implies that the degree of every vertex of R in the graph $R+S$ is $n-k+k=n$.
Thus the graph $R+S$ is n-regular graph. That is $R+S\in\Re$. As such S is extension set of R with respect to $\Re$. Hence $\underset{\Re}{ext}(R)\le k$.

Suppose that $\underset{\Re}{ext}(R)<k$. Then there exists an extension set $h=\{a_1,a_2,...,a_l\}$ of R with cardinality $l<k$ such that $R+h\in\Re$. By similar argument to part (1) above, we get that the degree of every vertex of h in the graph $R+h$ is n. That is $d(a_1)=d(a_2)=...=d(a_l)=n$ and the degree of every vertex of R in the graph $R+h$ is $r+l$. As $R+h$ is regular graph, we must have $n=r+l$ which implies $r=n-l$ with $l<k$ a contradiction to our assumption that $r=n-k$ and k is the smallest number of vertices. Hence $\underset{\Re}{ext}(R)=k$. □

Theorem 2.9: Let ε be the class of eulerian graphs, and $G\in\varepsilon$. Then

$$\underset{\varepsilon}{ext}(G)=\begin{cases} 2 & \text{if the order of G is even} \\ \infty & \text{if the order of G is odd} \end{cases}$$

Proof: Let ε be the class of eulerian graphs, and $G\in\varepsilon$. Then G is connected graph and the degree of every vertex in G is even Harary[1969].

Suppose that G has even order.

Let $S=\{v_1,v_2\}$ be an independent set of two vertices which are different from the vertices of G. Join every vertex of S to every vertex of G exactly one. Then, as the order of G is even, then every vertex of S in the graph $G+S$ has a degree even, and since S consists of two vertices, then the degree of every vertex of G in the graph $G+S$ increase by 2. Thus the degree of every vertex in the graph $G+S$ is even.

It is easy to prove that the graph $G+S$ is connected. Hence the graph $G+S$ is eulerian. As such S is extension set of G with respect to ε. Hence $\underset{\varepsilon}{ext}(G)\leq 2$.

Suppose that $\underset{\varepsilon}{ext}(G)\neq 2$. Then there exists an extension vertex v_0 of G such that $G+v_0\in\varepsilon$ which is impossible as in the graph $G+v_0$ every vertex in G has a degree odd. Then $G+v_0\notin\varepsilon$ and $\underset{\varepsilon}{ext}(G)\neq 1$. Hence $\underset{\varepsilon}{ext}(G)=2$.

Now, if G has odd order. Then every extension vertex of G has a degree odd. Thus for any extension set h of G, the graph $G+h\notin\varepsilon$. Hence, by definition 2.6, $\underset{\varepsilon}{ext}(G)=\infty$. □

3. Extensibility of Digraphs.

In this section, we introduced the concepts of extension of digraphs, extensible class of digraphs and the extensibility number of digraph. Further, we characterized the extensibility number of regular and eulerian digraphs.

Here we introduce the concept of extension of a digraph.

Definition 3.1: Let D be a nontrivial digraph. The *extension* of D is a digraph denoted by $D+S$ obtained from D by adding a nonempty set of independent vertices S such that every vertex in S is adjacent or adjacent by every vertex in D but not both . In such away S is called *extension set* of D. In particular, if S consists of a single element v, then v is called *extension vertex* of D.

The definition of *extensible class of digraphs* is analogous to that in definition 2.3, only replace every graph G by a digraph D.

Proposition 3.2:

1. the class of connected digraphs is extensible class.
2. the class of Hamiltonian digraphs is extensible class.

The proof is similar to the proof of proposition 2.4. □

Proposition 3.3:

1. the class of regular digraphs is not extensible class.
2. the class of eulerian digraphs is not extensible class.

The proof is similar to the proof of proposition 2.5 part (3) and (4) respectively. □

The definition of *extensibility number of digraph* is analogous to that in definition 2.6 only replace every graph G by a digraph D as following.

Definition 3.4: let $\Im$ be the class of digraphs with certain property, and $D\in\Im$ be a nontrivial. The *extensibility number* of D with respect to $\Im$ is the smallest positive integer m, if exists, such that there exists an extension set S of D with cardinality m in which the new digraph $D+S\in\Im$. We write $m=\underset{\Im}{ext}(D)$. If such a number does not exist for D, then we say the corresponding extensibility number is ∞.

Theorem 3.5: Let $\Re$ be the class of regular digraphs, D be an r-regular digraph with even number of vertices n in $\Re$. Then $\underset{\Re}{ext}(D)=k$ if and only if k is the smallest number of vertices and D has regularity degree $r=\frac{n-k}{2}$.

Proof: Let D be an r-regular digraph with even number of vertices n.
Suppose that $\underset{\Re}{ext}(D)=k$. Then by definition 3.4, there exists an extension set $S=\{v_1,v_2,...,v_k\}$ of D and k is the smallest cardinality of S such that the digraph $D+S\in\Re$. By definition 3.1, S is independent set of vertices and every vertex in S is adjacent or adjacent by but not both every vertex in D. As $D+S$ is regular digraph and D has even order n, then every vertex of S in the digraph $D+S$ has indegree $\frac{n}{2}$ and outdegree $\frac{n}{2}$. That is $id(v_1)=od(v_1)=\frac{n}{2}$, $id(v_2)=od(v_2)=\frac{n}{2}$, … , $id(v_k)=od(v_k)=\frac{n}{2}$.
Also every vertex of D in the digraph $D+S$ has indegree $r+\frac{k}{2}$ and outdegree $r+\frac{k}{2}$. As $D+S$ is regular digraph, then we must have $\frac{n}{2}=r+\frac{k}{2}$ which implies $r=\frac{n-k}{2}$.

Conversely, suppose that k is the smallest number of vertices and D has regularity degree $r=\frac{n-k}{2}$. We prove that $\underset{\Re}{ext}(D)=k$.
Let $S=\{v_1,v_2,...,v_k\}$ be a set of independent vertices with cardinality k and vertices different from the vertices of D. Let every vertex of S is adjacent to $\frac{n}{2}$ vertices of D and adjacent by the remaining $\frac{n}{2}$ vertices of D such that every vertex in D has indegree $r+\frac{k}{2}$ and outdegree $r+\frac{k}{2}$. As $r=\frac{n-k}{2}$, then every vertex of D in the digraph $D+S$ has indegree $\frac{n-k}{2}+\frac{k}{2}=\frac{n}{2}$ and outdegree $\frac{n}{2}$. Then the digraph $D+S$ is regular. As such S is extension set of D with respect to $\Re$. Hence $\underset{\Re}{ext}(D)\le k$.
If $\underset{\Re}{ext}(D)<k$, then there exists an extension set $h=\{u_1,u_2,...,u_l\}$ of D with cardinality $l<k$ such that $R+h\in\Re$. By similar argument to part (1) above, we get that the indegree of every vertex of h in the digraph $D+h$ is $\frac{n}{2}$ and the outdegree is $\frac{n}{2}$. Also the indegree of every vertex of D in the digraph $D+h$ is $r+\frac{l}{2}$ and the outdegree is $r+\frac{l}{2}$. Since the digraph $D+h$ is regular, then we must have $\frac{n}{2}=r+\frac{l}{2}$ which implies $r=\frac{n-l}{2}$ with $l<k$ a contradiction to our assumption that $r=\frac{n-k}{2}$ and k is the smallest number of vertices. Hence $\underset{\Re}{ext}(D)=k$.□

Theorem 3.6: Let ε be the class of eulerian digraphs, and $D\in\varepsilon$. Then

$$\underset{\varepsilon}{ext}(D)=\begin{cases}2 & \text{if the order of D is even}\\ \infty & \text{if the order of D is odd}\end{cases}$$

The proof is similar to the proof of theorem 2.9. □

3. Conclusions.

We conclude from this results that we can extend some graphs by adding vertices to get a new graphs with the same properties of there original graphs. We found some extensible classes of graphs and digraphs also the extensibility numbers for some graphs and digraphs.The authors can check the extensibility number for other kinds of graphs and digraphs.

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قـابــليـــة توسيـــــــع البيـــانات

اكـــرم بــرزان عطــــار

قســـم الريــاضيــــات - كليـــةالــتربيـــة - جـــامـــعـــة ذي قـــار

العراق – ذي قار

الملخص

في هذا البحث تم تقديم مفهوم توسيع البيانات ودراسة البيانات القابلة للتوسيع والغير قابلة للتوسيع. كذلك تم تقديم مفهوم عدد التوسيع للبيانات وإيجاد قيمته للبيانات المنتظمة وبيانات اويلر.