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# Bochner curvature tensor of Almost Kahler manifold

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## Abstract

In this paper we study the Bochner curvature tensor of almost Kahler manifold. We found the components of Bochner tensor of almost Kahler manifold in the adjoint G-structure space by using Kirichenko's tensors. It has been proved that an almost Kahler manifold is a manifold of class $\beta_1$ if and only if it is Kahler.

## 1. Introduction

The Bochner tensor was given by S. Bochner (1949)[4]. He found this tensor in the Kahler manifold as Weyle's tensor(conformal curvature of Riemannian manifold). S. Tachibana (1967)[13] gave it the real form and proved that the Bochner tensor has a meaning on any almost Hermatian manifold. M. Mastumoto(1969)[10] proved that Kahler manifold of constant scalar curvature tensor with zero Bochner tensor is local symmetric. S. Tachibana (1970)[14] proved that Kahler manifold of constant scalar curvature tensor with zero Bochner tensor is local holomorphic-isometric of product complex spaces. Z. Olsgak(1984)[12] gave classification of 4-dimensioal compact flat Bochner of Kahler manifold with non positive scalar curvature tensor. M. Petrovic and L. Verstraclen(1987)[11] are classified flat Bochner of Kahler manifold that the Weyle's tensor satisfies some conditions. A. Al-Othman(1993)[2] studied the Bochner tensor of Nearly Kahler manifold, he found the classification of flat Bochner tensor of NK-manifold and studied the Bochner tensor of B-constant type of almost Hermatian manifold and he defined the holomorphic Bochner curvature of almost Hermatian manifold. A. Al-Othman(2008)[1] studied the Bochner-recurrent Nearly Kahler manifold, he proved that Bochner-Recurrent Nearly Kahler manifold is either Bochner-symmetrical or Bochner-recurrent Kahler manifold.

In this present work we give the components of Bochner tensor of Almost Kahler manifold and study the almost Kahler manifold of class $\beta_1$ .

## 2. Preliminiries

Definition 1.2 [5]

A tensor field $J$ of type (1,1) is called an almost complex structure, such that, at each point $p \in M$ can be defined an endomorphism of the tangent space $T_p(M)$ with the property $J^2 = -id$, where $id: T_p(M) \to T_p(M)$ is the identity transformation.

Definition 2.2 [5]

A manifold provided by the almost complex structure is called an almost complex manifold.
It is well-known, that every complex manifold has even dimension and it is orintable. In general the converse is not true[8].

In the module $X^c(M)$ can be defined two projections $\sigma = \frac{1}{2}(id - J\sqrt{-1})$ and $\bar{\sigma} = \frac{1}{2}(id + J\sqrt{-1})$, where $X^c(M)$ is the complexification of the module $X^c(M)$.

The setting of projections $\sigma$ *and* $\bar{\sigma}$ is equivalent to the decomposition of the module $X^c(M)$ in the direct sum of these projections.
i.e. $\forall X \in X^c(M),\ X = \sigma(X) + \bar{\sigma}(X)$.

Definition 3.2 [8]

The pair $\{J, g = <.,.>\}$ is called an almost Hermitian structure ($AH$- structure) on the manifold $M$, where $J$ is the almost complex structure on $M$, $g = <.,.>$ is a Riemannian metric on $M$, such that $< X, Y > = < JX, JY >,\ X, Y \in X(M)$.

Definition 4.2 [8]

A manifold provided by $AH$- structure is called an almost Hermiation manifold. It is known [6] that the setting of an almost Hermiation structure on $M$ is equivalent to the sitting of adjoint G-structure on $M$ with structure group is a unitary group $U(n)$. This $G$-structure is called an adjont $G$-structure space[7].

Assume that the value of indices $a, b, c, d, e, g, h, \ldots$ is in the range 1 to n, and the indices $i, j, k, l, \ldots$ is in the range 1 to 2n. Denote $\hat{a} = a + n$ , then the indices are $a, b, c, d, e, f, g, \ldots, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{f}, \hat{g}, \ldots$ .

In the space of the adjoint G-structure, the components of the tensor fields $J$ *and* $g$ are given by the matrices:[9]

$$(g_{ij}) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},\ (J_i^j) = \begin{pmatrix} I_n\sqrt{-1} & 0 \\ 0 & -I_n\sqrt{-1} \end{pmatrix} \qquad (2.1)$$

Where $I_n$ is the unit matrix of order n.

Definition 5.2 [9]

An $AH$-structure is called an almost Kahler structure($AK$- structure) if the fundamental form $\Omega(X, Y) = < X, JY >$ is closed i.e. $d\,\Omega = 0$.

A manifold $M$ with $AK$-structure is called an almost Kahler manifold($AK$-manifold).

Definition 6.2 [9]

The components of the fundamental form in the adjoint G-structure space are given by the matrix:

$$\left(\Omega_{ij}\right) = \begin{pmatrix} 0 & I_n\sqrt{-1} \\ -I_n\sqrt{-1} & 0 \end{pmatrix} \qquad (2.2)$$

Definition 7.2 [9]

The Riemannian curvature tensor $R$ for $M$ is 4-covariant tensor:
$R: T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \to \mathbb{R}$ which is defined by:
$R(X_1, X_2, X_3, X_4) = g(R(X_3, X_4)X_2, X_1)$ where $X_i \in T_p(M) \quad \forall i = 1, \ldots, 4$ and satisfied the following properties :

1. $R(X_1, X_2, X_3, X_4) = -R(X_2, X_1, X_3, X_4)$

2. $R(X_1,X_2,X_3,X_4) = -R(X_1,X_2,X_4,X_3)$
3. $R(X_1,X_2,X_3,X_4) = R(X_2,X_1,X_4,X_3)$
4. $R(X_1,X_2,X_3,X_4) + R(X_1,X_3,X_4,X_2) + R(X_1,X_4,X_2,X_3) = 0$

Proposition 8.2 [9]

The components of Riemannian curvature tensor of *AK*-manifold in the adjont G-structure space are:

1. $R^{\hat{a}}_{\hat{b}\hat{c}d} = A^{bc}_{ad} + 2B^{bch}\ B_{had} - 4B_{dah}\ B^{cbh}$
2. $R^{a}_{\hat{b}\hat{c}d} = 4B^{cah}\ B_{dbh} - A^{ac}_{bd} - 2B^{ach}\ B_{hbd}$
3. $R^{a}_{bc\hat{d}} = A^{ad}_{bc} + 2B^{adh}\ B_{hbc} - 4B^{dah}\ B_{cbh}$
4. $R^{\hat{a}}_{\hat{b}c\hat{d}} = 4B^{dbh}\ B_{cah} - A^{bd}_{ac} - 2B^{bdh}\ B_{hac}$
5. $R^{\hat{a}}_{\hat{b}cd} = -2B^{b}_{acd}$
6. $R^{a}_{bcd} = 2B^{a}_{bcd}$

7. $R^{a}_{b\hat{c}\hat{d}} = 2B^{adc}_{b}$ 8. $R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}} = 2B^{bcd}{}_{a}$ 9. $R^{a}_{\hat{b}cd} = 4B^{hab}\ B_{hcd}$ 10. $R^{a}_{\hat{b}\hat{c}d} = -2B^{cab}_{d}$ 11. $R^{a}_{\hat{b}c\hat{d}} = 2B^{dab}_{c}$ 12. $R^{a}_{\hat{b}\hat{c}\hat{d}} = -4B^{[c|ab|d]}$ 13. $R^{\hat{a}}_{bcd} = -4B_{[c|ab|d]}$ 14. $R^{\hat{a}}_{b\hat{c}d} = 2B_{dab}{}^{c}$ 15. $R^{\hat{a}}_{bc\hat{d}} = -2B_{cab}{}^{d}$ 16. $R^{\hat{a}}_{b\hat{c}\hat{d}} = 4B^{hcd}\ B_{hab}$

## 3. Bochner curvature tensor

Definition 1.3 [2]

Bochner curvature tensor on *AH*-manifold defined as the following form:

$$\begin{aligned}\beta(X,Y,Z,W) &= R(X,Y,Z,W) + L(X,W)g(Y,Z) - L(X,Z)g(Y,W) + L(Y,Z)g(X,W) \\ &\quad - L(Y,W)g(X,Z) + L(JX,W)g(JY,Z) - L(JX,Z)g(JY,W) \\ &\quad + L(JY,Z)g(JX,W) - L(JY,W)g(JX,Z) - 2L(JX,Y)g(JZ,W) \\ &\quad - 2L(JZ,W)g(JX,Y)\ p\end{aligned}$$

Where $L(X,Y) = -\frac{1}{2n+4}\, g(rX,Y) + \frac{K}{2(2n+2)(2n+4)} g(X,Y)$, $r$ is a Ricci tensor and $K$ is a scalar curvature tensor and $X,Y,Z,W \in X(M)$.

Let $C(X,Y) = L(JX,Y)$ and we have $g(JX,Y) = -\Omega(X,Y)$, thus

$$\beta_{ijkl} = R_{ijkl} + L_{il}g_{jk} - L_{ik}g_{jl} + L_{jk}g_{il} - L_{jl}g_{ik} - C_{il}\Omega_{jk} + C_{ik}\Omega_{jl} - C_{jk}\Omega_{il} + C_{jl}\Omega_{ik} + 2C_{ij}\Omega_{kl} + 2C_{kl}\Omega_{ij}$$

where $L_{ij} = -\frac{1}{2n+4} r_{ij} + \tilde{K} g_{ij}$ (1.3)

and $\tilde{K} = \frac{K}{2(2n+2)(2n+4)}$

and $C_{ij} = L(Je_i, e_j) = -\frac{1}{2n+4} J_i^k r_{kj} + \tilde{K} J_i^k g_{kj}$ (2.3)

Theorem 1.3

The components of Bochner curvature tensor of $AK$-manifold are:

1. $\beta_{abcd} = R_{abcd}$
2. $\beta_{\hat{a}bcd} = R_{\hat{a}bcd} + \frac{1}{2n+4}(r_{bd}\,\delta_c^a - r_{bc}\,\delta_d^a) + \frac{1}{2n+4}(r_{cd}\,\delta_b^a - r_{ac}\,\delta_b^d - 2r_{ad}\,\delta_c^b)$
3. $\beta_{a\hat{b}cd} = R_{a\hat{b}cd} + \frac{1}{2n+4}(r_{ac}\,\delta_b^d - r_{ad}\,\delta_c^b) + \frac{1}{2n+4}(r_{dc}\,\delta_a^b - r_{cd}\,\delta_a^b - 2r_{ad}\,\delta_c^b)$
4. $\beta_{ab\hat{c}d} = R_{ab\hat{c}d} + \frac{1}{2n+4}(r_{bd}\,\delta_a^c - r_{ad}\,\delta_b^c) + \frac{1}{2n+4}(2r_{cb}\,\delta_a^d - r_{cd}\,\delta_a^b - r_{cb}\delta_d^a)$
5. $\beta_{abc\hat{d}} = R_{abc\hat{d}} + \frac{1}{2n+4}(r_{ac}\delta_b^d - r_{bc}\delta_a^d) + \frac{1}{2n+4}(2r_{cb}\delta_a^d - r_{bc}\delta_a^d - r_{ac}\delta_b^d)$
6. $\beta_{\hat{a}\hat{b}cd} = R_{\hat{a}\hat{b}cd}$
7. $\beta_{\hat{a}b\hat{c}d} = R_{\hat{a}b\hat{c}d} - \frac{1}{n+2}(r_d^a\delta_b^c + r_b^a\delta_d^c + r_b^c\delta_d^a + r_d^c\delta_b^a) + 4\tilde{K}\delta_{db}^{ac}$
8. $\beta_{\hat{a}bc\hat{d}} = R_{\hat{a}bc\hat{d}} + \frac{1}{n+2}(r_c^a\delta_b^d + r_b^a\delta_c^d + r_b^d\delta_c^a + r_c^d\delta_b^a) - 4\tilde{K}\delta_{cb}^{ad}$

Proof

suppose that $M$ is $AK$-manifold, in the adjoint $G$-structure space by using (2.1), (2.2), (1.3) and (2.3) we get :

1. put $i = a, j = b$ we obtained $L_{ab} = -\frac{1}{2n+4} r_{ab}$ (3.3)

2.put $i = \hat{a}, j = \hat{b}$ we get $L_{\hat{a}\hat{b}} = -\frac{1}{2n+4} r_{\hat{a}\hat{b}}$ (4.3)

3. put $i = \hat{a}, j = b$ we get $L_{\hat{a}b} = -\frac{1}{2n+4} r_b^a + \hat{K}\,\delta_b^a$ (5.3)

4.put $i = a, j = \hat{b}$ we get $L_{a\hat{b}} = -\frac{1}{2n+4} r_a^b + \hat{K}\,\delta_a^b$ (6.3)

We compute the components of $C_{ij}$, in the same computes we obtained:

$C_{ab} = -\frac{\sqrt{-1}}{2n+4}\, r_{cb}\,\delta_a^c$ (7.3)

$C_{\hat{a}\hat{b}} = \frac{\sqrt{-1}}{2n+4}\, r_{\hat{c}\hat{b}}\,\delta_c^a$ (8.3)

$C_{\hat{a}b} = \frac{\sqrt{-1}}{2n+4}\, r_b^l\,\delta_l^a - \sqrt{-1}\,\tilde{K}\,\delta_b^a$ (9.3)

$C_{a\hat{b}} = -\frac{\sqrt{-1}}{2n+4}\, r_a^b\,\delta_a^b + \sqrt{-1}\,\tilde{K}\,\delta_a^b$ (10.3)

Now we compute the components of Bochner curvature tensor:

1.put $i = a, j = b, k = c, l = d$ then:

$$\beta_{abcd} = R_{abcd} + L_{ad}\,g_{bc} - L_{ac}\,g_{bd} + L_{bc}\,g_{ad} - L_{bd}\,g_{ac} - C_{ad}\,\Omega_{bc} + C_{ac}\,\Omega_{bd} - C_{bc}\,\Omega_{ad} + C_{bd}\,\Omega_{ac} + 2C_{ab}\,\Omega_{cd} + 2C_{cd}\,\Omega_{ab}$$

From equations (2.1), (2.2), (1.3) – (10.3) we get :

$\beta_{abcd} = R_{abcd}$ (11.3)

2. put $i=\hat{a}, j=b, k=c, l=d$ then:

$$\beta_{\hat{a}bcd}=R_{\hat{a}bcd}+L_{\hat{a}d}\,g_{bc}-L_{\hat{a}c}\,g_{bd}+L_{bc}\,g_{\hat{a}d}-L_{bd}\,g_{\hat{a}c}-C_{\hat{a}d}\,\Omega_{bc}+C_{\hat{a}c}\,\Omega_{bd}-C_{bc}\,\Omega_{\hat{a}d}+C_{bd}\,\Omega_{\hat{a}c}+2C_{\hat{a}b}\,\Omega_{cd}+2C_{cd}\,\Omega_{\hat{a}b}$$

From equations (2.1, 2.2, 1.3-10.3) we get :

$$\beta_{\hat{a}bcd}=R_{\hat{a}bcd}+\frac{1}{2n+4}\left(r_{bd}\,\delta_c^a-r_{bc}\,\delta_d^a\right)+\frac{1}{2n+4}\left(r_{cd}\,\delta_b^a-r_{ac}\,\delta_b^d-2r_{ad}\,\delta_c^b\right) \quad (12.3)$$

3. put $i=a, j=\hat{b}, k=c, l=d$ then:

$$\beta_{a\hat{b}cd}=R_{a\hat{b}cd}+\frac{1}{2n+4}\left(r_{ac}\,\delta_b^d-r_{ad}\,\delta_c^b\right)+\frac{1}{2n+4}\left(r_{dc}\,\delta_a^b-r_{cd}\,\delta_a^b-2r_{ad}\,\delta_c^b\right) \quad (13.3)$$

4. put $i=a, j=b, k=\hat{c}, l=d$ then:

$$\beta_{ab\hat{c}d}=R_{ab\hat{c}d}+\frac{1}{2n+4}\left(r_{bd}\,\delta_a^c-r_{ad}\,\delta_b^c\right)+\frac{1}{2n+4}\left(2\,r_{cb}\,\delta_a^d-r_{cd}\,\delta_a^b-r_{cb}\,\delta_d^a\right) \quad (14.3)$$

5. put $i=a, j=b, k=c, l=\hat{d}$ then:

$$\beta_{abc\hat{d}}=R_{abc\hat{d}}+\frac{1}{2n+4}\left(r_{ac}\delta_b^d-r_{bc}\delta_a^d\right)+\frac{1}{2n+4}\left(2r_{cb}\delta_a^d-r_{bc}\delta_a^d-r_{ac}\delta_b^d\right) \quad (15.3)$$

6.put $i=\hat{a}, j=\hat{b}, k=c, l=d$ then:

$$\beta_{\hat{a}\hat{b}cd}=R_{\hat{a}\hat{b}cd} \quad (16.3)$$

7. put $i=\hat{a}, j=b, k=\hat{c}, l=d$ then:

$$\beta_{\hat{a}b\hat{c}d}=R_{\hat{a}b\hat{c}d}-\frac{1}{n+2}\left(r_d^a\delta_b^c+r_b^a\delta_d^c+r_b^c\delta_d^a+r_d^c\delta_b^a\right)+4\tilde{K}\delta_{db}^{ac} \quad (17.3)$$

8. put $i=\hat{a}, j=b, k=c, l=\hat{d}$ then:

$$\beta_{\hat{a}bc\hat{d}}=R_{\hat{a}bc\hat{d}}+\frac{1}{n+2}\left(r_c^a\delta_b^d+r_b^a\delta_c^d+r_b^d\delta_c^a+r_c^d\delta_b^a\right)-4\tilde{K}\delta_{cb}^{ad} \quad (18.3)$$

Definition 2.3 [2]

The Bochner curvature tensor is of:

1. class $\beta_1$ if $\beta(X,Y,Z,W)=\beta(X,Y,JZX,JW)$
2. class $\beta_2$ if $\beta(X,Y,Z,W)=\beta(JX,JY,Z,W)+\beta(JX,Y,JZ,W)+\beta(JX,Y,Z,JW)$
3. class $\beta_3$ if $\beta(X,Y,Z,W)=\beta(JX,JY,JZX,JW)$

Definition 3.3 [3]

An *AH*-manifolds is called a Kahler manifold if $B^{abc}=0$ and called an almost kahler manifold if $B^{(abc)}=0$ where $B^{abc}=0$ is structure tensor( Kirichenko's tensor), and the bracket ( ) denote to symmetric.

Theorem 2.3

Almost Kahler manifold $M$ is of class $\beta_1$ if and only if $M$ is Kahler manifold.

Proof

According to class $\beta_1$ we get :

$$\begin{aligned}\beta_{\hat{a}\hat{b}cd}&=\beta(\varepsilon_{\hat{a}},\varepsilon_{\hat{b}},\varepsilon_c,\varepsilon_d)=\beta(\varepsilon_{\hat{a}},\varepsilon_{\hat{b}},J\varepsilon_c,J\varepsilon_d)\\&=\beta\left(\varepsilon_{\hat{a}},\varepsilon_{\hat{b}},\sqrt{-1}\varepsilon_c,\sqrt{-1}\varepsilon_d\right)\\&=\left(\sqrt{-1}\right)\left(\sqrt{-1}\right)\beta(\varepsilon_{\hat{a}},\varepsilon_{\hat{b}},\varepsilon_c,\varepsilon_d)\\&=-\beta(\varepsilon_{\hat{a}},\varepsilon_{\hat{b}},\varepsilon_c,\varepsilon_d)=-\beta_{\hat{a}\hat{b}cd}\end{aligned}$$

Thus $\beta_{\hat{a}\hat{b}cd}+\beta_{\hat{a}\hat{b}cd}=0 \Rightarrow \beta_{\hat{a}\hat{b}cd}=0$

Since $\beta_{\hat{a}\hat{b}cd} = 4B^{hab}B_{hcd} \implies B^{hab}B_{hcd} = 0$

By folding (a and c) and ( b and d) we get

$B^{hab}B_{hab} = 0 \Leftrightarrow \sum|B_{hab}|^2 = 0 \Leftrightarrow B_{hab} = 0$

According to [3]this Kahler condition.

Therefore $M$ is Kahler manifold.

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## تنسر انحناء بوخنر لمتعدد الطيات كوهلر التقريبي

**الخلاصة**


في هذا البحث تم حساب مركبات تنسر انحناء بوخنر لمتعدد الطيات كوهلر التقريبي واثبات أن متعدد الطيات كوهلر التقريبي يكون من الصنف $\beta_1$ إذا وفقط إذا كان من الصنف كوهلر.