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On Pre-connected sets in Bitopological Spaces

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Abstract

In this paper,we give and study a generalization for the concept "pre-connected set" in bitopological spaces , through as well as we give several various results related to this generalization .

Introduction

The study of pre-connected sets in Bitopological Space takes place at several sources and researchers like Kelly (1963) , Valiru (1977) , Jelles (1989) , Al-Swidi (1993) and Dontcher (1998) .A generalization of the definition of the pre-connected set is presented with some theorems and examples to explain and illustrative it .Conclusions are added at the end of it.

1. Some definitions

Definition (1) :[2]
Let (X,τ) **be a topological space. A subset** A **of** X **is said to be** $pre-$**open set iff** $A \subseteq \operatorname{int}_\tau(cl_\tau(A))$. **The family of all** $pre-$**open sets is denoted by** $pr-O(X)$.

Definition (2) :[3]
Let (X,τ,ρ) **be a bitopological space. A subset** A **of** X **is called pre-open set with respect to the two topologies** τ,ρ **if** $A \subseteq \operatorname{int}_\tau\lfloor cl_\rho(A)\rfloor$.
The collection of all pre-open sets with respect to the two topologies τ **and** ρ **is denoted by** $pr-O(X)$.

Definition (3) :[2]
Let (X,τ,ρ) **be a bitopological space . A subset** A **of** X **is called** $pre-$ **closed set of** X **iff the complement of** A **is** $pre-$**open set of** X. **The collection of all** $pr-$**closed sets with respect to the two topologies** τ **and** ρ **denoted by** $pr-C(X)$.

Note (1) :
The family of all $pr-open$ **sets of** X **is not necessary a topology on** X, **see the following example .**

Example (1) :
Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\}\}$, **and** $\rho=\{X,\phi,\{c\}\}$. (X,τ), (X,ρ) **are two topological spaces.Then** (X,τ,ρ) **is bitopological space , so :**

$$pr-O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\},$$

Definition (4) :[6]
Let (X,τ,ρ) be a bitopological space.Two non- empty subsets A and B of X are said to be $pr-$separated iff
$$[A \cap pr-cl(B)] \cup [B \cap pr-cl(A)] = \phi$$
That is A and B are $pr-$separated sets iff A and B are disjoint and every one contains no limit points of the other with respected to $pr-O(X)$.

Example (2) :
Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{a,b\}\}$ and $\rho=\{X,\phi,\{c\},\{a,b\}\}$. (X,τ), (X,ρ) are two topological spaces, then (X,τ,ρ) is a bitopological space , so , $pr-O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$,
Let $A=\{a\}$ and $B=\{b\}$ are two $pr-$open sets.
Hence A and B are $pr-$separated sets , since
$pr-cl(A)=pr-cl(\{a\})=\{a\}, pr-cl(B)=pr-(\{b\})=\{b\}$,
and $[A \cap pr-cl(B)] \cup [pr-cl(A) \cap B] =$
$[\{a\} \cap \{b\}] \cup [\{a\} \cap \{b\}] = \phi \cup \phi = \phi.$
So , A and B are $pr-$separated subsets of X

Definition (5) :[6]
Let (X,τ,ρ) be a bitopological space. A subset A of X is said to be $pr-$disconnected set iff it is the union of two $pr-$separated non-empty sets in X , that is $\exists$ two non-empty sets C and D in X such that
$pr-cl(C) \cap D = \phi$,
$C \cap pr-cl(D) = \phi$ **and** $A = C \cup D$ **.**
A is called $pr-$connected iff it is not $pr-$disconnected set .

Example (3) :
Let
$X=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{b\},\{a,b\}\}$ **and** $\rho=\{X,\phi,\{b\},\{a,c\}\}$. **$(X,\tau)$, (X,ρ) are two topological spaces. Then (X,τ,ρ) is a bitopological space , so ,**
$pr-O(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{b,c\}\}$,

Let $A=\{a,b\}$ and $C=\{a\}$, $D=\{b\}$ so $A=C \cup D$.
Hence $pr-cl(C)=\{a\}, pr-cl(D)=\{b\}$,
$[pr-cl(C) \cap D] \cup [C \cap pr-cl(D)] = [\{a\} \cap \{b\}] \cup [\{a\} \cap \{b\}] = \phi$.
Therefore A is $pr-$disconnected set , but if $B=\{b,c\}, C=\{b\}, D=\{b,c\}$
hence $pr-cl(C)=\{b\}, pr-cl(D)=\{b,c\}$
$[pr-cl(C) \cap D] \cup [C \cap pr-cl(D)] = [\{b\} \cap \{b,c\}] \cup [\{b\} \cap \{b,c\}] = \{b\} \cup \{b\} = \{b\} \neq \phi$
Then B is $pr-$connected set .

Notes (2) :
1. **The empty set is trivially $pr-$connected .**
2. **Every singleton $pr-$open set is $pr-$connected set , since it can not be expressed as a union of two non-empty $pr-$separated sets .**

Definition (6) :[6]
Two points a,b of a bitopological space (X,τ,ρ) are said to be $pr-$connected iff they are contained in a $pr-$connected subset of X .

Definition (7) :[4]
Let (X,τ,ρ) be a bitopological space , and Y be a subset of X . The relative bitopological space for Y is denoted by (Y,τ_Y,ρ_Y) , such that :
$\tau_Y = \{G \cap Y : G \in \tau\}$
$\rho_Y = \{H \cap Y : H \in \rho\}$
(Y,τ_Y,ρ_Y) is called a subspace of the bitopological space (X,τ,ρ), the relative bitopological space for Y with respect to $pr-$open sets is the collection $pr-O(X)_Y$ given by
$pr-O(X)_Y = \{G \cap Y : G \in pr-O(X)\}$

2. Various results :

Theorem (1) :
Let (Y,τ_Y,ρ_Y) **be subspace of** (X,τ,ρ) **let** $A,B\subset pr-O(X)$ **then** A **and** B **are** $pr-O(X)-$**separated iff they** $pr-O(X)_Y$ **-separated.**

Theorem (2) :
Let (Y,τ_Y,ρ_Y) **be a subspace of bitopological** (X,τ,ρ) **and** $A\subset Y$**.Then** A **is** $pr-$**disconnected iff it is** pr_Y **-disconnected .**

Proof :
By above theorem, therefore, A **is the union of two** $pr-$**separated sets iff it is the union of two** pr_Y **-separated sets .**

Theorem (3) :
A bitopological space (X,τ,ρ) **is** $pr-$**disconnected iff** $\exists$ **a non-empty proper subset of** X **which is both** pr**-open and** $pr-$**closed in** X **.**

Proof :

$\Leftarrow$

Let A **a non-empty proper subset of** X **which both** pr**-open and** $pr-$**closed.We will show that** X **is a** $pr-$**disconnected .**

Let $B=A^C\Rightarrow B\neq\phi$ **, since** $A\subset X$ **, so** $B\bigcup A=X$ **and** $A\bigcap B=\phi$ **…… (*)**
Since A **is both** pr**-open and** $pr-$**closed** $\Rightarrow B$ **is both** pr**-open and** $pr-$**closed , hence** $pr-cl(A)=A,\ pr-cl(B)=B.$
Thus X **is** $pr-$**disconnected .**

$\Rightarrow$

Let X **is** $pr-$**disconnected ,Then** $\exists$ **two non-empty** $pr-$**separated subsets** A **and** $B\ni X=A\bigcup B$ **…………….(**)**
$pr-cl(A)\bigcap B=\phi$ **and** $A\bigcap pr-cl(B)=\phi$ **,**
since $A\subset pr-cl(A)$ **and** $pr-cl(A)\bigcap B=\phi\Rightarrow A\bigcap B=\phi$**.**

Hence , $A=B^C$ **, by (**) ,** $B\neq\phi\Rightarrow X\subseteq A\bigcup pr-cl(B)$ **since** $(B\subset pr-cl(B))$**.**
But $A\bigcup pr-cl(B)\subseteq X\Rightarrow X=A\bigcup pr-cl(B)$ **, also** $A\bigcap pr-cl(B)=\phi\Rightarrow A=[pr-cl(B)]^c$**.**

Similarity , $B=[pr-cl(A)]^C$**. Since** $pr-cl(A)$ **and** $pr-cl(B)$ **are** $pr-$**closed sets, therefore** A **and** B **are** $pr-$**open sets, and hence** $A=B^C$ **is a** $pr-$**closed set (since** $A\bigcap B=\phi$**) .Thus** A **is a non-empty** $pr-$**closed and** $pr-$**open set .**

Theorem (4) :
Let (X,τ,ρ) **be a bitopological space.Then** X **is a** $pr-$**disconnected iff any one of the following statements holds :**
i. X **is the union of two non-empty disjoint** $pr-$**open sets .**
ii. X **is the union of two non-empty disjoint** $pr-$ **closed sets .**

Proof :

$\Rightarrow$

Let X **be a** $pr-$**disconnected** $\Rightarrow\exists A\neq\phi, A\subset X$ **which is both** $pr-$**closed and** $pr-$**open set .Therefore** A^C **is also both** $pr-$**closed and** $pr-$**open set and** $A\bigcup A^C=X$ **.**

Hence the sets A and A^c are satisfy (i) and (ii) .

$\Leftarrow$

Let $X = A \bigcup B$ and $A \bigcap B = \phi$, where A and B are non-empty pr-open ($pr-$closed) sets $\Rightarrow A = B^C$,so that A is pr-closed($pr-$open)set.

Since $B \neq \phi, B \subset X \Rightarrow A \neq \phi, A \subset X$ which is both pr-open and $pr-$closed .

Hence X is a $pr-$disconnected .

Theorem (5) :

Let E be a $pr-$connected subset of a bitopological space (X, τ, ρ).If $F \subset X$ such that $E \subset F \subset pr-cl(E)$, then F is a $pr-$connected . In particular , $pr-cl(E)$ is $pr-$connected .

Proof :

Suppose that F is $pr-$disconnected

$\Rightarrow \exists A, B \neq \phi, A \bigcap pr-cl(B) = \phi, pr-cl(A) \bigcap B = \phi, A \bigcup B = F$, since

$E \subset F = A \bigcup B \Rightarrow E \subset A$ or $E \subset B$.

Let $E \subset A \Rightarrow pr-cl(E) \subset pr-cl(A) \Rightarrow pr-cl(E) \bigcap B \subset pr-cl(A) \bigcap B = \phi \Rightarrow$

$pr-cl(E) \bigcap B = \phi$..........................(*)

Also , $A \bigcup B = F \subset pr-cl(E) \Rightarrow B \subset F \subset pr-cl(E) \Rightarrow$

$pr-cl(E) \bigcap B = B$....................(**)

By (*) and (**) we get $B = \phi$ which is a contradiction , since $B \neq \phi$.

Hence F is a $pr-$connected . Again , since $E \subset pr-cl(E) \subseteq pr-cl(E) \Rightarrow$

$pr-cl(E)$ is a $pr-$connected .

Theorem (6) :

Let (X, τ, ρ) be a bitopological space ,and E is a subset of X .If each two points of E are $pr-$connected in some $pr-$connected subset of E , then E is a $pr-$connected subset of X .

Proof :

Suppose that E is not $pr-$connected , then $\exists A, B \neq \phi$ and $A, B \subset X \ni A \bigcap pr-cl(B) = \phi$, and $pr-cl(A) \bigcap B = \phi, E = A \bigcup B$, since $A, B \neq \phi \Rightarrow \exists a \in A$ and $b \in B$ such that a and b must be contained in some $pr-$connected subset F of E since $F \subset A \bigcup B \Rightarrow$ either $F \subset A$ or $F \subset B \Rightarrow$

either $a, b \in A$ or $a, b \in B$. Let $a, b \in A$, since $b \in B \Rightarrow A \bigcap B \neq \phi$, which is a contradiction, since A and B are disjoint . Hence E must be a $pr-$connected .

Theorem (7) :

Let $\{G_\lambda : \lambda \in \wedge\}$ be a family of $pr-$connected subsets of a bitopological space (X, τ, ρ) such that $\bigcap \{G_\lambda : \lambda \in \wedge\} \neq \phi$ .Then $\bigcup \{G_\lambda : \lambda \in \wedge\}$ is a $pr-$connected set .

Proof :

Suppose that $E = \bigcup \{G_\lambda : \lambda \in \wedge\}$ is a $pr-$disconnected $\Rightarrow \exists$ two non-empty disjoint sets G_1 and G_2 both $pr-$open in the subset E of $X \ni E = G_1 \bigcup G_2$,

$\forall \lambda, G_1 \bigcap G_\lambda, G_2 \bigcap G_\lambda$ are disjoint sets both $pr-$open in the subset $G_\lambda \ni$

$(G_1 \bigcap G_\lambda) \bigcup (G_2 \bigcap G_\lambda) = (G_1 \bigcup G_2) \bigcap G_\lambda = G_\lambda$

………………..(*)

Since G_λ is a $pr-$connected , one of the sets $G_1 \bigcap G_\lambda$ and $G_2 \bigcap G_\lambda$ must be empty , say $G_1 \bigcap G_\lambda = \phi$, then by (*) $G_2 \bigcap G_\lambda = G_\lambda \Rightarrow G_\lambda \subset G_2$.

$\forall \lambda \in \wedge \Rightarrow \bigcup \{G_\lambda : \lambda \in \wedge\} \subset G_2 \Rightarrow G_1 \bigcup G_2 \subset G_2 \Rightarrow G_1 = \phi$

, which is contradiction .

Hence E **must be** $pr-$**connected since** G_1 **is non-empty , so** E **is** $pr-$**connected .**

Theorem (8) :

Let $\{G_\lambda : \lambda \in \wedge\}$ **be a family of** $pr-$**connected subsets of a bitopological space (** X, τ, ρ **) such that one of the member of this family intersects every other member , then** $\bigcup_{\lambda \in \wedge} G_\lambda$ **is a** $pr-$**connected set .**

Proof :

Let $G_{\lambda \circ}$ **be a fixed member of the given family** $\ni G_{\lambda \circ} \bigcap G_\lambda \neq \phi, \forall \lambda \in \wedge$ **then** $D_\lambda = G_{\lambda \circ} \bigcup G_\lambda$ **is a** $pr-$**connected set** $\forall \lambda \in \wedge$ **by previous theorem .**

Now ,

$\bigcup \{D_\lambda : \lambda \in \wedge\} = \bigcup \{G_{\lambda \circ} \bigcup G_\lambda : \lambda \in \wedge\} =$
$G_{\lambda \circ} \bigcup [\bigcup \{G_\lambda : \lambda \in \wedge\}] = \bigcup \{G_\lambda : \lambda \in \wedge\}$

and

$\bigcap \{D_\lambda : \lambda \in \wedge\} = \bigcap \{G_{\lambda \circ} \bigcup G_\lambda : \lambda \in \wedge\} =$
$\bigcap \{D_\lambda : \lambda \in \wedge\} = G_{\lambda \circ} \bigcup [\bigcap \{G_\lambda : \lambda \in \wedge\}] \neq \phi$

(always) ,since by our assumption $G_{\lambda \circ} \neq \phi$ **is intersects every** $G_\lambda \neq \phi$ **as** $G_{\lambda \circ} \bigcap G_\lambda \neq \phi, \forall \lambda \in \wedge$**.**

Hence by previous theorem $\bigcup \{D_\lambda : \lambda \in \wedge\} = \bigcup \{G_\lambda : \lambda \in \wedge\}$ **is a** $pr-$**connected set .**

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في هذا البحث نقدم وندرس تعميم لمفهوم ً المجموعة المتصلة تقريباً في الفضاءات ثنائية التبولوجي من خلال تقديمنا عدة نتائج متنوعة التي تتعلق بهذا التعميم .