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Some Properties of Regular Line Graphs

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Abstract

In this paper, the concept of *regular line graph* has been introduced. The maximum number of vertices with different degrees in the regular line graphs has also been studied. Further, the necessary and sufficient condition for regular line graph to be bipartite graph have also been proved.

Key words: Line Graphs, Regular graphs, Connected graphs, Bipartite Graphs.

1. Introduction

A *graph* $G=(V(G),E(G))$ consists of two finite sets, $V(G)$, the *vertex set* of the graph, often denoted by just V, which is a nonempty set of elements called *vertices*, and E(G), the *edge set* of the graph, often denoted by just E, which is a possibly empty set of elements called *edges*, such that each edge e in E is assigned an unordered pair of vertices (u,v) called the *end vertices* of e.The number of vertices of G will be called the *order* of G, and will usually be denoted by p; the number of edges of G will generally be denoted by q. If for a graph G, $p=1$ then G is called *trivial graph*; if $q=0$ then G is called a *null graph*. We shall usually denote the edge corresponding to (v,w) where (v and w are vertices of G) by vw.

If e is an edge of G having end vertices v,w then e is said to *join* the vertices v and w, and these vertices are then said to be *adjacent*. In this case, we also say that e is *incident* to v and w, and that w is a *neighbor* of v. The *open neighborhood* $N(v)$ of the vertex v consists of the set of vertices adjacent to v, that is $N(v)=\{w\in V: vw\in E\}$. An *independent set of vertices* in G is a set of vertices of G no two of which are adjacent. If two distinct edges are incident with a common vertex, then they are *adjacent edges*. An *independent set of edges* in G is a set of edges of G no two of which are adjacent.

Let v be a vertex of the graph G. If v joined to itself by an edge, such an edge is called *loop*. The degree $d(v)$ is the number of edges of G incident with v, counting each loop twice. If two (or more) edges of G have the same end vertices then these edges are called *parallel*. A graph is called *simple* if it has no loops and parallel edges. We say that G is *regular graph* with regularity degree r if the degree of every vertex is r.

A simple graph in which every two vertices are adjacent is called a *complete graph*; the complete graph with p vertices is denoted by K_p. A *bipartite graph* is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that each edge of G joins V_1 with V_2. If G contains every edge join V_1 and V_2, then G is *complete bipartite*. If V_1 and V_2 have m and n vertices, we write $G=K_{m,n}$. A *star* is a complete bipartite graph $K_{1,n}$.

A *subgraph* of the *graph* $G=(V(G),E(G))$ is a graph $H=(V(H),E(H))$ such that $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$.

A *walk* in a graph G is a finite sequence $W=v_0e_1v_1\ e_2...v_{k-1}e_kv_k$ whose terms are alternatively vertices and edges such that for $1\le i\le k$, the edge e_i has ends v_{i-1} and v_i. The vertex v_0 is called the *origin* of the walk W, while v_k is called the *terminus* of W. The vertices $v_1,...,v_{k-1}$ in the above walk W are called *internal vertices*. If the edges $e_1,e_2,...,e_k$ of the walk $W=v_0e_1v_1\ e_2...v_{k-1}e_kv_k$ are distinct then W is called a *trail* and if $v_0=v_k$ then W is called a *closed trail*. If the vertices $v_0,v_1,...,v_k$ of the walk $W=v_0e_1v_1\ e_2...v_{k-1}e_kv_k$ are distinct then W is called a *path*. A path with n vertices will sometime be denoted by P_n. A closed trail in a graph G is called a *cycle* if its origin and internal vertices are distinct. A cycle with n vertices, will sometime be denoted by C_n and called n-cycle.

A graph G is *connected* if there is a path joining each pair of vertices of G; a graph which is not connected is called *disconnected*. A connected graph which contains no cycle is called a *tree*. A graph G is *Hamiltonian* if it has a cycle which includes every vertex of G.

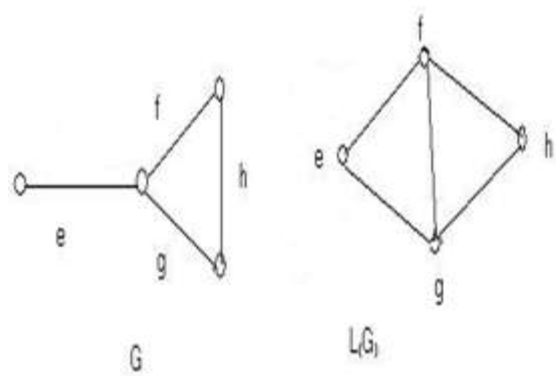
For the undefined concepts and terminology we refer the reader to Wilson[1978], Clark[1991], Harary[1969], West[1999] and Tutte[1984].

All graphs throughout this paper are simple.

2. Regular Line Graph

We need the following definition[2].

Definition 2.1: Let G be a simple graph, the line graph of G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef\in E(L(G))$ when e and f have a common endpoint in G.



If $e=uv$ is an edge of G, then the degree of e in $L(G)$ is clearly $d(u)+d(v)-2$.

Now, we define the regular line graph.

Definition 2.2: Let G be a nontrivial **(**non null) graph, if the line graph $L(G)$ is regular graph, then we call G is *regular line graph*. In particular if G is connected graph, we say that G is *connected regular line graph.*

Examples 1:

1-Every cycle graph C_n is connected regular line graph**.**

2-Every regular graph is regular line graph.

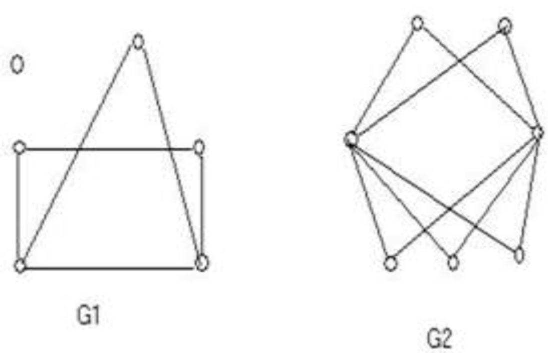


Figure 1

3- The star graph is connected regular line graph**.**

4- Each of the two graphs in Figure 1 is regular line graph.

We introduce the following theorem.

Theorem 2.3: Every connected regular line graph has at most two vertices with different degrees.

Proof: Let G be a connected regular line graph and the regularity degree of $L(G)$ is r. Suppose that u is a vertex of degree d_1 in G. As $L(G)$ is regular, all the vertices in the neighbors of u have the same degree. Let v be a vertex in $N(u)$ with degree d_2. Assume that $d_1 \neq d_2$ and x is a vertex in G different from u,v with degree d_3 such that $d_3 \neq d_1$ and $d_3 \neq d_2$. Then all the vertices in $N(x)$ have the same degree. Let y be a vertex in $N(x)$ of degree d_4.

Now, as $L(G)$ is regular graph with regularity degree r, and from Definition 2.1, we have

$d_1 + d_2 - 2 = r$ **…** (1)

$d_3 + d_4 - 2 = r$ **…** (2)

From (1) and (2), we get $d_1 + d_2 = d_3 + d_4$.

If $d_1 = d_4$, then $d_2 = d_3$ a contradiction.

If $d_2 = d_4$, then $d_1 = d_3$ a contradiction.

Hence $d_4 \neq d_1$ and $d_4 \neq d_2$.

As G is connected graph, there exist a path from u to x. In this path either a vertex of degree d_1 or a vertex of degree d_2 is adjacent to a vertex of degree d_4.

Suppose that a vertex of degree d_1 is adjacent to a vertex of degree d_4. As $L(G)$ is regular, we have $d_1 + d_4 - 2 = r$. By using (1), we get $d_2 = d_4$ a contradiction. A similar contradiction occur when a vertex of degree d_2 is adjacent to a vertex of degree d_4. Therefore $d_1 + d_4 - 2 \neq r$ and also $d_2 + d_4 \neq r$ which is a contradiction to our choice of G. Hence G has at most two vertices with different degrees.

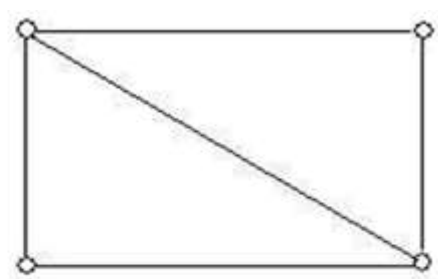


Figure 2

The converse of this theorem need not be true. In fact, the graph in Figure 2 has at most two vertices with different degrees, but it is clear that the graph is not connected regular line graph.

Theorem 2.4: Let G be a connected regular line graph, then G is bipartite graph if and only if one of the following holds.

1. G contains two vertices with different degrees. Or
2. G is regular graph and isomorphic to K_2 or all it is cycles are even.

Proof: Suppose that G is bipartite graph. As G is a connected regular line graph, by Theorem 2.3, G has at most two vertices with different degrees. That is either G has two vertices with different degrees and (1) holds, or all the vertices in G have the same degree, and in this case G is regular. If the regularity degree of G is 1, then G is isomorphic to K_2. If the regularity degree of G is greater than 1, then G contains some cycle [If every cycle of a graph G has degree at least 2, then G contains a cycle. West p.27], and as G is bipartite, then every cycle in G is even [A graph is bipartite iff all it is cycles are even. Harary p.18] and (2) holds.

Conversely, suppose that (1) holds.

As G is a connected regular line graph, by Theorem 2.3, the vertices in $V(G)$ have exactly two different degrees.

Let V_1 be a subset of all vertices in G in which each vertex has a degree d_1 ; V_2 be a subset of all vertices in G in which each vertex has a degree d_2 such that $d_1 \neq d_2$ and $V_1 \bigcup V_2 = V$.

As G is connected regular line graph, there exist two adjacent vertices $u \in V_1$ and $v \in V_2$. As $d(u) = d_1$ and $d(v) = d_2$, the regularity degree of $L(G)$ is

$r = d_1 + d_2 - 2$ **…** (1)

Now , if the subset of vertices V_1 contains two adjacent vertices, then $r = d_1 + d_1 - 2$.

By using (1), we get $d_1 = d_2$ which is a contradiction to our assumption. Therefore, V_1 does not contain any two adjacent vertices. By Similar way we prove that V_2 does not contain any two adjacent vertices. Hence G is bipartite graph.

Suppose that (2) holds.

If G is regular and isomorphic to K_2 , then it is clear that G is bipartite. If all the cycles of G are even, then G is bipartite [A graph is bipartite iff all it is cycles are even. Harary p.18].

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بعض خواص البيانات ذات البيان الخطي المنتظم

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الملخـص

في هذا البحث قدمنا مفهوم البيانـات ذات البيـان الخطـي المنـتظم. ثـم دراسنا بعض خواص تلك البيانـات. حيث تـم اثبات ان البيانـات ذات البيان الخطي المنـتظم تحـوي علـى الاكثر راسين بـدرجات مختلفـة. كمـا تـم ايجاد الشرط الضروري والكافي لها كي تكون بيانات ثنائية التجزئـة.