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Lyapunov-Schmidt Method in Bifurcation Solutions of Nonlinear Fourth Order Differential Equation

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Abstract

This paper studied the bifurcation solutions of elastic beams equation by using Lyapunov-Schmidt method. The bifurcation equation corresponding to the elastic beams equation has been found. Also, the Discriminant set (bifurcation set) of the elastic beams equation has been found for some values of parameters.

Key Words: Bifurcation solutions, Local Lyapunov-Schmidt method, Bifurcation set.

1. Introduction

It is known that many of the nonlinear problems that appear in Mathematics and Physics can be written in the form of operator equation,

$$f(x,\lambda)=b, \quad x\in O\subset X, \quad b\in Y, \quad \lambda\in R^n. \qquad \dots (1.1)$$

where *f* is a smooth Fredholm map of index zero and X, Y are Banach spaces and O is an open subset of X. For these problems, the method of reduction to finite dimensional equation,

$$\Theta(\xi,\lambda,\varepsilon)=\beta, \quad \xi\in\hat{M}, \quad \beta\in\hat{N}. \qquad \dots (1.2)$$

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Can be used, where $\hat{M}$ and $\hat{N}$ are smooth finite dimensional manifolds.

Passage from equation (1.1) into the equation (1.2) (Variant local scheme of Lyapunov-Schmidt) with the conditions, that equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, *etc*) dealing with [3],[13],[14],[10] .

The oscillations and motion of waves of the elastic beams on elastic foundations can be described by means of the following PDE,

$$\frac{\partial^2 y}{\partial t^2}+\frac{\partial^4 y}{\partial x^4}+\alpha\frac{\partial^2 y}{\partial x^2}+\beta\, y+yy^{//}=\psi\ ,$$

where y is the deflection of beam and $\psi=\varepsilon\,\varphi(x)$ (ε - small parameter) is a continuous function. It is known that, to study the oscillations of beams, equilibrium state ($w(x)=y(x,t)$) should be considered which is described by the equation,

$$\frac{d^4 w}{d\,x^4}+\alpha\frac{d^2 w}{d\,x^2}+\beta\, w+ww^{//}=\psi, \qquad \dots\ (1.3)$$

Equation (1.3) is a special case of the general nonlinear differential equation of fourth order,

$$\begin{gathered}\frac{d^4 w}{d\,x^4}+\alpha\frac{d^2 w}{d\,x^2}+\beta\, w+g(\lambda,\tilde{w})=\psi,\\ \tilde{w}=(w,w^{/},w^{//},w^{///},w^{////}).\end{gathered} \qquad \dots\ (1.4)$$

When $g(\lambda,\tilde{w})=-k\,w^3$ and $\psi=0$ equation (1.4) has been studied as follows: Thompson and Stewart [6] showed numerically the existence of periodic solutions of equation (1.4) for some values of parameters. Bardin and Furta [1] used the local method of Lyapunov-Schmidt and found the sufficient conditions of existence of periodic waves of equation (1.4), also they introduced the solutions of equation (1.4) in the form of power series. Furta and Piccione [5] showed the existence of periodic travelling wave solutions of equation (1.4) describing oscillations of an infinite beam, which lies on a non-linearly elastic support with non-small amplitudes. Sapronov [12] applied the local method of Lyapunov –Schmidt and found the bifurcation solutions of equation (1.4) when $g(\lambda,\tilde{w})=w^3$ and $\psi=0$ with the boundary conditions,

$$w(0)=w(\pi)=w^{//}(0)=w^{//}(\pi)=0.$$

In his study he solved the bifurcation equation corresponding to the equation (1.4) and found the bifurcation diagram of a specify problem. When $g(\lambda,\tilde{w}) = w^2$ and $\psi \neq 0$, equation (1.4) has been studied by Abdul Hussain [7], it was shown that by using local method of Lyapunov – Schmidt the existence of bifurcation solutions of equation (1.4) with the conditions,

$$w(0) = w(1) = w^{//}(0) = w^{//}(1) = 0.$$

and another study with the conditions,

$$\begin{aligned} & w(x_1) \geq \varepsilon_1, \quad w(x_2) \geq \varepsilon_2, \\ & 0 < x_1 < x_2 < 1, \quad \varepsilon_1, \varepsilon_2 \text{ are small parametrs.} \end{aligned}$$

Also, a new study of corner singularities of smooth maps was given in the analysis of bifurcations balance of the elastic beams and periodic waves. When $g(\lambda,\tilde{w}) = w^2 + w^3$ equation (1.4) was studied by Sapronov [2], in his work he found bifurcation periodic solutions of equation (1.4) by using local method of Lyapunov –Schmidt. Also, he solved the bifurcation equation corresponding to the equation (1.4) and found the bifurcation diagram of a specify problem. When $g(\lambda,\tilde{w}) = w^2 + w^3$ and $\psi \neq 0$ equation (1.4) was studied by Mohammed [9], in his work he found bifurcation solutions of equation (1.4) by using local method of Lyapunov –Schmidt. Also, he solved the bifurcation equation corresponding to the equation (1.4) and found the bifurcation diagram of a specify problem. When $g(\lambda,\tilde{w}) = w^2 + w^3$, $\psi \neq 0$ and $\varepsilon_1, \varepsilon_2 \neq 0$ equation (1.4) was studied by Abdul Hussain (2009) [8], he found the bifurcation solutions of equation (1.4) by using local method of Lyapunov-Schmid

In this paper we used local method of Lyapunov –Schmidt to study the bifurcation solutions of the boundary value problem,

$$\begin{aligned} & \frac{d^4 w}{d x^4} + \alpha \frac{d^2 w}{d x^2} + \beta w + w w^{//} = \psi, \\ & w(0) = w(\pi) = w^{//}(0) = w^{//}(\pi) = 0. \end{aligned} \quad \text{…. (1.5)}$$

We shall introduce two basic definitions.

Definition 1.1[4] suppose that *E* and *F* are Banach spaces and $A: E \to F$ be a linear continuous operator. The operator *A* is called Fredholm operator, if

1- The kernel of *A*, Ker(*A*), is finite dimensional,
2- The rang of *A*, Im(*A*), is closed in *F*,
3- The Cokernel of A, Coker (A), is finite dimensional.

The number *dim*(*Ker A*)- *dim*(*Coker A*)
is called Fredholm index of the operator *A* and denote it by *ind*(*A*).

Definition 1.2[12] The set of all λ in which equation (1.1) has degenerate solutions is called the Discriminant set.

2. Reduction to the bifurcation equation

To study problem (1.5) it is convenient to set the ODE in the form of operator equation [12], that is;

$$f(w,\lambda)=\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+\beta w+w w^{//} \qquad \ldots (2.1)$$

where $f: E\rightarrow M$ is a nonlinear Fredholm map of index zero from Banach space E to Banach space M, $E=C^4([0,\pi],R)$ is the space of all continuous functions that have derivative of order at most four, $M=C^0([0,\pi],R)$ is the space of all continuous functions and $w=w(x),\;\; x\in[0,\pi],\; \lambda=(\alpha,\beta)$. In this case the bifurcation solutions of equation (2.1) is equivalent to the bifurcation solutions of the operator equation,

$$f(w,\lambda)=\psi, \qquad \psi\in M\,. \qquad \ldots (2.2)$$

The first step in this reduction is to determine the linearized equation corresponding to the equation (2.2), which is given by the following equation,

$$Ah=0, \quad h\in E,$$

$$A=\frac{\partial f}{\partial w}(0,\lambda)=\frac{d^4}{dx^4}+\alpha\frac{d^2}{dx^2}+\beta,$$

$$h(0)=h(\pi)=h^{//}(0)=h^{//}(\pi)=0.$$

The solutions of linearized equation is given by,

$$e_p(x)=c_p\sin(px), \qquad p=1,2,3,...$$

and the characteristic equation corresponding to this solution is,

$$p^4-\alpha p^2+\beta=0.$$

This equation gives in the $\alpha\beta$-plane characteristic lines ℓ_p. The characteristic lines ℓ_p consist of the points (α,β) in which the linearized equation has non-zero solutions. The point of intersection of characteristic lines in the $\alpha\beta$-plane is a bifurcation point [12]. So for equation (2.1) the point $(\alpha,\beta)=(5,4)$ is a bifurcation point. Localized parameters α, β as follows,

$$\alpha=5+\delta_1, \quad \beta=4+\delta_2, \qquad \delta_1, \delta_2 \text{ are small.}$$

Lead to bifurcation along the modes $e_1(x)=c_1\sin(x)$, $e_2(x)=c_2\sin(2x)$, where $\|e_1\|=\|e_2\|=1$, and $c_1=c_2=\sqrt{\frac{2}{\pi}}$.

Let $N = Ker(A) = span\ \{e_1, e_2\}$, then the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E=N\oplus N^{\perp},\qquad N^{\perp}=\{v\in E: v\perp N\}.$$

Similarly, the space M can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$M=N\oplus \tilde{N}^{\perp},\qquad \tilde{N}^{\perp}=\{v\in M: v\perp N\}.$$

There exist two projections $P: E\to N$ and $(I-P): E\to N^{\perp}$ such that

$$Pw=u,\qquad (I-P)w=v$$

and hence every vector $w\in E$ can be written in the form,

$$w=u+v,\qquad u=\sum_{i=1}^{2}\xi_i\, e_i\in N,\qquad v\in N^{\perp},\qquad \xi_i=\langle w, e_i\rangle.$$

Similarly, there exist two projections $Q: M\to N$ and $(I-Q): M\to \tilde{N}^{\perp}$ such that

$$QF(w,\lambda)=F_1(w,\lambda),$$
$$(I-Q)F(w,\lambda)=F_2(w,\lambda)$$

Every vector $w\in E$ can be written in the form,

$$w=u+v,\qquad u=\sum_{i=1}^{2}\xi_i\, e_i\in N,\qquad N\perp v\in E^{\infty-2},\qquad \xi_i=\langle w, e_i\rangle.$$

and hence,

$$F(w,\lambda)=F_1(w,\lambda)+F_2(w,\lambda),$$
$$F_1(w,\lambda)=\sum_{i=1}^{2}v_i(w,\lambda)e_i\in N,\qquad F_2(w,\lambda)\in\tilde{N}^{\perp},$$
$$v_i(w,\lambda)=\langle F(w,\lambda),e_i\rangle.$$

Since $\psi\in M$ implies that $\psi=\psi_1+\psi_2$, $\psi_1=t_1e_1+t_2e_2\in N$, $\psi_2\in\tilde{N}^{\perp}$. Accordingly, equation (2.2) can be written in the form,

$$QF(w,\lambda)=\psi_1,$$
$$(I-Q)F(w,\lambda)=\psi_2$$

Or

$$QF(u+v,\lambda)=\psi_1,$$
$$(I-Q)F(u+v,\lambda)=\psi_2$$

By implicit function theorem, there exists a smooth map $\Phi : N \to N^{\perp}$ (depending on λ), such that, $\Phi(u,\lambda)=v$ and

$$(I-Q)F(u+\Phi(u,\lambda),\lambda)=\psi_2 .$$

To find the solutions of the equation $F(w,\lambda)=\psi$ in the neighbourhood of the point $w=0$ it is sufficient to find the solutions of the equation,

$$QF(u+\Phi(u,\lambda),\lambda)=\psi_1. \quad \ldots (2.3)$$

Equation (2.3) is called bifurcation equation of the equation (2.1) and then we have the bifurcation equation in the form,

$$\Theta(\xi,\lambda)=\psi_1, \quad \xi=(\xi_1,\xi_2), \quad \lambda=(\alpha,\beta),$$

Where,

$$\Theta(\xi,\lambda)=F_1(u+\Phi(u,\lambda),\lambda).$$

Equation (2.1) can be written in the form,

$$F(u+v,\lambda)=A(u+v)+T(u+v)$$
$$=Au+uu^{//}+\cdots$$

Where, $T(u+v)=(u+v)(u^{//}+v^{//})$ and the dots denote the terms consisting the element *v*.

Hence

$$\Theta(\xi,\lambda)=F_1(u+v,\lambda)$$
$$=\sum_{i=1}^{2}\langle Au+uu^{//},e_i\rangle e_i+\cdots=\psi_1. \quad \ldots (2.4)$$

Where ($\langle\cdot,\cdot\rangle_H$ is the scalar product in Hilbert space L_2 ([0, π], *R*)). Equation (2.4) implies that,

$$\sum_{i=1}^{2}\langle Au+uu^{//},e_i\rangle e_i+\cdots=t_1 e_1+t_2 e_2 \quad \ldots (2.5)$$

After some calculations of equation (2.5) we have the following result,

$$(A_1\xi_1^2+A_2\xi_2^2+A_3\xi_1)e_1+(B_1\xi_1\xi_2+B_2\xi_2)e_2=t_1 e_1+t_2 e_2$$

Where,

$A_1=\frac{-8}{3\pi}\sqrt{\frac{2}{\pi}}$, $A_2=\frac{16}{5}A_1$, $A_3=\tilde{\alpha}_1(\lambda)$, $B_1=4A_1$, $B_2=\tilde{\alpha}_2(\lambda)$, and $\tilde{\alpha}_1(\lambda),\tilde{\alpha}_2(\lambda)$ are spectral smooth functions. The symmetry of the function $\psi(x)$ with respect to the involution $I:\psi(x)\mapsto\psi(\pi-x)$ implies that t_2=0 and then we have stated the following result,

THEOREM 2.1 The bifurcation equation

$$\Theta(\xi,\lambda)=F_1(u+\Phi(u,\lambda),\lambda)=\psi_1$$

Corresponding to the equation (2.2) has the following form,

$$\Theta(\xi,\tilde{\lambda})=\begin{pmatrix}\xi_1^2+\frac{16}{5}\xi_2^2+\lambda_1\xi_1+q_1\\ \frac{5}{4}\xi_1\xi_2+\lambda_2\xi_2\end{pmatrix}+o(|\xi|^2)+O(|\xi|^2)O(\delta)=0\,.$$

Where, $\xi=(\xi_1,\xi_2),\quad \tilde{\lambda}=(\lambda_1,\lambda_2,q_1)\in R^3,\quad \delta=(\delta_1,\delta_2).$ □

The discriminant set Σ of the map $\Theta(\xi,\tilde{\lambda})$ is locally equivalent in the neighbourhood of point zero to the discriminant set of the map

$\Theta_1(\xi,\tilde{\lambda})$ [11],

$$\Theta_1(\xi,\tilde{\lambda})=\begin{pmatrix}\xi_1^2+\frac{16}{5}\xi_2^2+\lambda_1\xi_1+q_1\\ \frac{5}{4}\xi_1\xi_2+\lambda_2\xi_2\end{pmatrix} \qquad \text{…. (2.6)}$$

this means that, to study the discriminant set of the map $\Theta(\xi,\tilde{\lambda})$ it is sufficient to study the discriminant set of the map $\Theta_1(\xi,\tilde{\lambda})$. The point $a\in E$ is a solution of equation (2.2) if and only if

$$a=\sum_{i=1}^{2}\bar{\xi}_i e_i+\Phi(\bar{\xi},\bar{\lambda}),$$

Where, $\bar{\xi}$ is a solution of equation

$$\Theta_1(\xi,\tilde{\lambda})=0. \qquad \text{… (2.7)}$$

In many applications the Discriminant set (Caustic) can be solved by finding a relationship between the parameters and variables given in the problem, but in some problems there is a difficulty for finding this parameterization. The second way for finding the Discriminant set it is by finding the parameter equation, that is; equation of the form

$$h(\tilde{\lambda})=0, \quad \tilde{\lambda}=(\lambda_1,\lambda_2,q_1)\in R^3.$$

such that the set of all $\tilde{\lambda}=(\lambda_1,\lambda_2,q_1)$ in which equation (2.7) has degenerate solutions that satisfy the equation $h(\tilde{\lambda})=0$,where $h: R^3 \to R$ is a map. For equation (2.7) it is not easy to find a relationship between the parameters and variables given in the equation, so the second way has been used to determine the Discriminant set of equation (2.7). Let

$$z=\frac{2}{5}\lambda_1\lambda_2+q_1, \quad s=\frac{4}{5}\lambda_1+\lambda_2 .$$
$$and$$
$$e=\frac{-4}{5}\lambda_2$$

Then the following result has been stated,

THEOREM 2.2 Caustic (The discriminant) of equation (2.7) in the space of parameters $(\lambda_1,\lambda_2,q_1)$ is a union of the following two surfaces

1) $z^2-\lambda_1 zs-q_1 s^2=0,$
2) $s_1^{\,2}-2\lambda_1\lambda_2-q_1=0.$

The last theorem can be proved by solving the following two systems in terms of q_1 , λ_1, λ_2 . The systems are,

$$\left.\begin{aligned}&\xi_2=0,\\&\xi_1^2+\frac{16}{5}\xi_2^2+\lambda_1\xi_1+q_1=0,\\&\xi_1^2+\frac{16}{5}\xi_2^2+(\frac{1}{2}\lambda_1+\frac{4}{5}\lambda_2)\xi_1+\frac{2}{5}\lambda_1\lambda_2=0.\end{aligned}\right)$$

$$\left.\begin{aligned}&\xi_1=-\frac{4}{5}\lambda_2,\\&\xi_1^2+\frac{16}{5}\xi_2^2+\lambda_1\xi_1+q_1=0,\\&\xi_1^2+\frac{16}{5}\xi_2^2+(\frac{1}{2}\lambda_1+\frac{4}{5}\lambda_2)\xi_1+\frac{2}{5}\lambda_1\lambda_2=0.\end{aligned}\right)$$

To study the Discriminant set of equation (2.7) it is convenient to fix the value of one parameters. The sections of Discriminant set in some planes were described in the figures (1), (2), (3), (4) and (5). (All figures were found by using Maple 11).

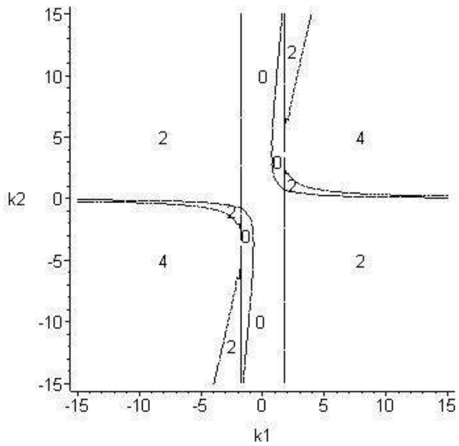

Fig. 1 describes caustic of equation (2.6) when $q1_{>0}$

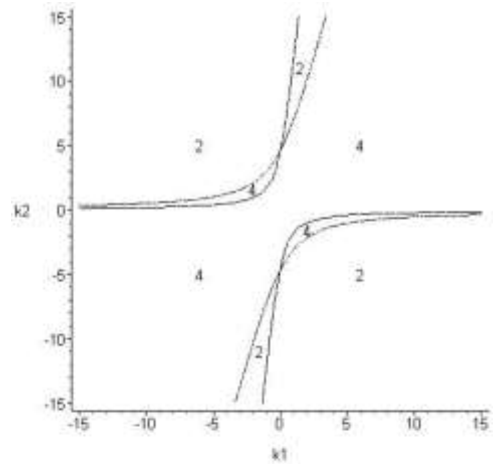

Fig. 2 describes caustic of equation (2.6) when $q1_{<0}$

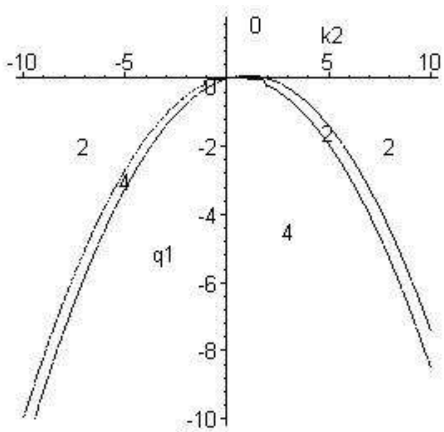

Fig. 3 describes caustic of equation (2.6) when $k1_{>0}$

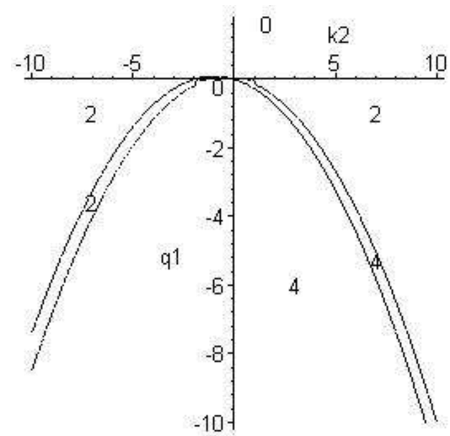

Fig. 4 describes caustic of equation (2.6) when $k1<0$

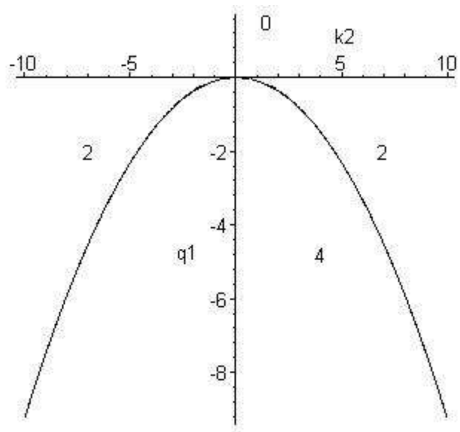

Fig. 5 describes caustic of equation (2.6)
When k1=0

Figure (1) describes the discriminant set in the $\lambda_1\lambda_2$-plane when q1>0. The number of regular solutions in every region was found is either zero, two or four. In Figure (2) the discriminant set has been found for q1<0 in the $\lambda_1\lambda_2$-plane and the number of regular solutions in every region is two or four. Figure (3), (4) and (5) describes the discriminant set in the $\lambda_2 q_1$-plane when k1>0, k1< 0 and k1=0 respectively. The number of regular solutions in every region was found is either zero, two or four.

In figure (1) the complement of the Discriminant set $\Gamma = R^2 \setminus \Sigma$ is the union of three open subsets $\Gamma = \Gamma_0 \bigcup \Gamma_2 \bigcup \Gamma_4$ such that if Γ_0 have no regular solutions, if Γ_2 have two regular solutions with topological indices 1, -1 and if Γ_4 have four regular solutions with topological indices 1,-1,1,-1. In figure (2) the complement of the Discriminant set is a union of two open subsets $\Gamma = \Gamma_2 \bigcup \Gamma_4$ such that if Γ_2 have two regular solutions with topological indices 1, -1 and if Γ_4 have four regular solutions with topological indices 1,-1,1,-1. In figures (3), (4) and (5) the complement of the Discriminant set is the union of three open subsets $\Gamma = \Gamma_0 \bigcup \Gamma_2 \bigcup \Gamma_4$ such that if Γ_0 have no regular solutions, if Γ_2 have two regular solutions with topological indices 1, -1 and if Γ_4 have four regular solutions with topological indices 1,-1,1,-1.

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طريقة ليابونوف – شمدت لإيجاد حلول التفرع لمعادلة تفاضلية غير خطية من الرتبة الرابعة

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الخلاصة

دُرسَ في هذا البحث حلول التفرع لمعادلة الأنابيب المرنة باستخدام طريقة ليابونوف - شمدت . تم إيجاد معادلة التفرع المطابقة لمعادلة الأنابيب المرنة, كذلك تم إيجاد المجموعة المميزة(مجموعة التفرع) لمعادلة الأنابيب المرنة لبعض قيم المعلمات.